323 HONORS SPRING 2014, HOMEWORK DUE 3/4

These problems are in addition to the assigned problems from Leon.
Given a vector space V, let $V^* := \{f : V \to \mathbb{R} \mid f \text{ is linear}\}$ denote the *dual vector space* to V.

(1) Let $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ be column vectors, so w^T (the transpose of w) is a row vector. Show that the matrix $vw^T \in Mat_{n\times m}$ has rank one.
(2) Show that a nonzero matrix $X \in Mat_{n\times m}$ has rank one if and only if the determinants of all 2×2 submatrices of X are zero.
(3) Show that the rank of a matrix $X \in Mat_{n\times m}$ is the smallest number r such that X is the sum of r rank one matrices.
(4) Show that a matrix $X \in Mat_{n\times m}$ has rank at most r if and only if the determinants of all $(r+1) \times (r+1)$ submatrices of X are zero.
(5) Given nonzero elements $\alpha \in V^*$ and $w \in W$, form a linear transformation $\alpha \otimes w : V \to W$ by $(\alpha \otimes w)(v) := \alpha(v)w$. What is the image of this linear transformation? What is its rank?
(6) Show that if V is finite dimensional, $\dim V = \dim V^*$.

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