ON THE RANKS AND BORDER RANKS OF SYMMETRIC TENSORS

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Abstract. Motivated by questions arising in signal processing, computational complexity, and other areas, we study the ranks and border ranks of symmetric tensors using geometric methods. We provide improved lower bounds for the rank of a symmetric tensor (i.e., a homogeneous polynomial) obtained by considering the singularities of the hypersurface defined by the polynomial. We obtain normal forms for polynomials of border rank up to five, and compute or bound the ranks of several classes of polynomials, including monomials, the determinant, and the permanent.

1. Introduction

Let $S^d\mathbb{C}^n$ denote the space of complex homogeneous polynomials of degree d in n variables. The *rank* (or *Waring rank*) $R(\phi)$ of a polynomial $\phi \in S^d\mathbb{C}^n$ is the smallest number r such that ϕ is expressible as a sum of r d-th powers, $\phi = x_1^d + \cdots + x_r^d$ with $x_j \in \mathbb{C}^n$. The *border rank* $\underline{R}(\phi)$ of ϕ, is the smallest r such that ϕ is in the Zariski closure of the set of polynomials of rank r in $S^d\mathbb{C}^n$, so in particular $R(\phi) \geq \underline{R}(\phi)$. Although our perspective is geometric, we delay the introduction of geometric language in order to first state our results in a manner more accessible to engineers and complexity theorists.

Border ranks of polynomials have been studied extensively, dating at least back to Terracini, although many questions important for applications to enginering and algebraic complexity theory are still open. For example, in applications, one would like to be able to explicitly compute the ranks and border ranks of polynomials. In the case of border rank, this could be done if one had equations for the variety of polynomials of border rank r. Some equations have been known for nearly a hundred years: given a polynomial $\phi \in S^d\mathbb{C}^n$, we may *polarize* it and consider it as a multi-linear form $\tilde{\phi}$, where $\phi(x) = \tilde{\phi}(x, \ldots, x)$. We can then feed $\tilde{\phi}$ s vectors, to consider it as a linear map $\phi_{s,d-s} : S^s(\mathbb{C}^n)^* \to S^{d-s}(\mathbb{C}^n)$, where $\phi_{s,d-s}(x_1 \cdots x_s)(y_1 \cdots y_{d-s}) = \tilde{\phi}(x_1, \ldots, x_s, y_1, \ldots, y_{d-s})$. Macaulay [15] showed that for all $1 \leq s \leq d$,

$$\underline{R}(\phi) \geq \operatorname{rank}\phi_{s,d-s}. \tag{1}$$

See Remark 6.5 for a proof. These equations are sometimes called *minors of Catalecticant matrices* or *minors of symmetric flattenings.*

One important class of polynomials in applications are the monomials. We apply the above equations, combined with techniques from differential geometry to prove:

Theorem 1.1. *Let $a_0, \ldots, a_m$ be non-negative integers satisfying $a_0 \geq a_1 + \cdots + a_m$. Then*

$$\underline{R}(x_0^{a_0} x_1^{a_1} \cdots x_m^{a_m}) = \prod_{i=1}^{m} (1 + a_i).$$

For other monomials we give upper and lower bounds on the border rank, see Theorem 11.2.

We also use differential-geometric methods to determine normal forms for polynomials of border rank at most five and estimate their ranks, Theorems 10.2, 10.5, 10.6. For example:

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Theorem 1.2. *The polynomials of border rank three have the following normal forms:*

normal form	R
$x^d+y^d+z^d$	3
$x^{d-1}y+z^d$	$d\leq R\leq d+1$
$x^{d-2}y^2+x^{d-1}z$	$d\leq R\leq 2d-1$

Here one must account for the additional cases where x,y,z are linearly dependent, but in these cases one can normalize, e.g. $z=x+y$. More information is given in Theorem 10.2.

To obtain new bounds on rank, we use algebraic geometry, more specifically the singularities of the hypersurface determined by a polynomial ϕ. Let $\mathrm{Zeros}(\phi)=\{[x]\in\mathbb{PC}^{n*}\mid \phi(x)=0\}\subset\mathbb{PC}^{n*}$ denote the zero set of ϕ. Let $x_1,\dots,x_n$ be linear coordinates on $\mathbb{C}^{n*}$ and define

$$\Sigma_s(\phi):=\left\{[x]\in\mathrm{Zeros}(\phi)\;\middle|\;\frac{\partial^I\phi}{\partial x^I}(x)=0,\forall|I|\leq s\right\}$$

so $\Sigma_0(\phi)=\mathrm{Zeros}(\phi)$ and $\Sigma_1(\phi)$ is the set of singular points of $\mathrm{Zeros}(\phi)$.

While the following result is quite modest, we remark that it is the first new general lower bound on rank that we are aware of in nearly 100 years (since Macaulay's bound (1)):

Theorem 1.3. *Let $\phi\in S^dW$ with $\langle\phi\rangle=W$. Let $1\leq s\leq d$. Use the convention that $\dim\emptyset=-1$. Then,*

$$R(\phi)\geq\operatorname{rank}\phi_{s,d-s}+\dim\Sigma_s(\phi)+1.$$

The right hand side of the inequality is typically maximized at $s=\lfloor d/2\rfloor$, see §3.1.

For example, applying Theorem 1.3 to the determinant and permanent polynomials (see §9) yields

Corollary 1.4.

$$R(\det_n)\geq\binom{n}{\lfloor n/2\rfloor}+n^2-(\lfloor n/2\rfloor+1)^2,$$
$$R(\mathrm{perm}_n)\geq\binom{n}{\lfloor n/2\rfloor}+n(n-\lfloor n/2\rfloor-1).$$

Gurvits [8] had previously observed $\underline{R}(\det_n)\geq\binom{n}{\lfloor n/2\rfloor}$ and $\underline{R}(\mathrm{perm}_n)\geq\binom{n}{\lfloor n/2\rfloor}$ by using (1).

We expect that further study of singularities will produce significantly stronger general lower bounds for rank, including bounds that involve the degree as well as the number of variables.

As a consequence of our study of rank in a more general geometric context, we prove

Corollary 1.5. *Given $\phi\in S^d\mathbb{C}^n$, $R(\phi)\leq\binom{n+d-1}{d}+1-n$,*

which is a corollary of Proposition 5.1. (The bound $R(\phi)\leq\binom{n+d-1}{d}+1$ is trivial, as we explain in §5.)

1.1. **Overview.** We begin in §2 by phrasing the problems in geometric language. We then review standard facts about rank and border rank in §3. In §4 we give an exposition of a theorem of Comas and Seiguer [4], which completely describes the possible ranks of homogeneous polynomials in two variables. We then discuss ranks for arbitrary varieties and prove Proposition 5.1, which gives an upper bound for rank valid for an arbitrary variety in §5. Applying Proposition 5.1 to polynomials yields Corollary 1.5 above. In §6 we prove Theorem 1.3. In §7 we give a presentation of the possible ranks, border ranks and normal forms of degree three polynomials in three variables that slightly refines the presentation in [5]. We then study a few specific cubic polynomials in an arbitrary number of variables in §8, followed by a brief discussion of bounds on rank and border rank for determinants and permanents in §9. In §10 we return to a general study of limiting secant planes and use this to classify polynomials of border ranks up to five. We conclude with a study of the ranks and border ranks of monomials in §11.

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2. Geometric definitions

Definitions of rank and border rank in a more general context are as follows: Let $V = \mathbb{C}^n$ denote a complex vector space and let $\mathbb{P}V$ denote the associated projective space. For a subset $Z \subset \mathbb{P}V$, we let $\langle Z \rangle \subseteq V$ denote its linear span. For a variety $X \subset \mathbb{P}V$, define

$$\sigma_r^0(X) = \left\{ \bigcup_{x_1,\dots,x_r \in X} \mathbb{P}\langle x_1,\dots,x_r\rangle \right\}, \qquad \sigma_r(X) = \overline{\left\{ \bigcup_{x_1,\dots,x_r \in X} \mathbb{P}\langle x_1,\dots,x_r\rangle \right\}} \tag{2}$$

where the overline denotes Zariski closure. These are respectively the points that lie on some secant $\mathbb{P}^{r-1}$ to X and the Zariski closure of the set of such points, called the *variety of secant* $\mathbb{P}^{r-1}$*'s to* X. For $p \in \mathbb{P}V$, define the X*-rank* of p, $R_X(p) := \{\min r \mid p \in \sigma_r^0(X)\}$ and the X*-border rank* of p, $\underline{R}_X(p) := \{\min r \mid p \in \sigma_r(X)\}$. In geometry, it is more natural to study border rank than rank, because by definition the set of points of border rank at most r is an algebraic variety. Let S^dW denote the space of homogeneous polynomials of degree d on W^* and let $v_d(\mathbb{P}W) \subset \mathbb{P}(S^dW)$ denote the *Veronese variety*, the (projectivization of the) set of d-th powers. Then, comparing with the definitions of §1, $R(\phi) = R_{v_d(\mathbb{P}W)}([\phi])$ and $\underline{R}(\phi) = \underline{R}_{v_d(\mathbb{P}W)}([\phi])$. Advantages of the more general definitions include that it is often easier to prove statements in the context of an arbitrary variety, and that one can simultaneously study the ranks of polynomials and tensors (as well as other related objects). We also let $\tau(X) \subset \mathbb{P}V$ denote the variety of embedded tangent $\mathbb{P}^1$'s to X, called the *tangential variety of* X and note that $\tau(X) \subseteq \sigma_2(X)$.

At first glance, the set of polynomials (respectively points in $\mathbb{P}V$) of a given rank (resp. X-rank) appears to lack interesting geometric structure—it can have several components of varying dimensions and fail to be a closed projective variety. One principle of this paper is that *among polynomials of a given border rank, say* r_0*, the polynomials having rank greater than* r_0 *can be distinguished by their singularities.* For a hypersurface $X \subset \mathbb{P}V$ and $x \in X$, define $\mathrm{mult}_x(X)$ to be the minimum of the orders of vanishing of the defining equations for X at x.

Consider the following stratification of $\mathbb{P}S^dW$. Let

$$v_d(\mathbb{P}W^*)_k{}^\vee := \mathbb{P}\{\phi \in S^dW \mid \exists [p] \in \mathrm{Zeros}(\phi),\ \mathrm{mult}_{[p]}(\mathrm{Zeros}(\phi)) \geq k+1\}.$$

Then

$$\mathbb{P}S^dW = v_d(\mathbb{P}W^*)_0{}^\vee \supset v_d(\mathbb{P}W^*)^\vee = v_d(\mathbb{P}W^*)_1{}^\vee \supset \cdots \supset v_d(\mathbb{P}W^*)_d{}^\vee = \emptyset.$$

Among polynomials of a given border rank, we expect the deeper they lie in this stratification, the higher their rank will be. (The analogous stratification of $\mathbb{P}V^*$ can be defined for arbitrary varieties $X \subset \mathbb{P}V$. It begins with $\mathbb{P}V^*$ and the next stratum is $X^\vee$.) A first step in this direction is Theorem 1.3. We expect the general study of points whose X-rank is greater than their X-border rank will be closely related to stratifications of dual varieties.

3. Review of known facts about rank and border rank of polynomials

3.1. **The Alexander-Hirschowitz Theorem.** The expected dimension of $\sigma_r(X^n) \subset \mathbb{P}^N$ is $\min\{r(n+1)-1, N\}$, and if $\sigma_r(X)$ fails to have this expected dimension it is called *degenerate*. Alexander and Hirschowitz [1], building on work of Terracini, showed that the varieties $\sigma_r(v_d(\mathbb{P}W))$ are all of the expected dimensions with a short, well understood, list of exceptions, thus the rank and border rank of a generic polynomial of degree d in n variables is known for all d, n. (Note that it is essential to be working over an algebraically closed field to talk about a generic polynomial.) See [2] for an excellent exposition of the Alexander-Hirschowitz theorem.

3.2. **Subspace varieties.** Given $\phi \in S^dW$, define the *span* of ϕ to be $\langle\phi\rangle = \{\alpha \in W^* \mid \alpha \lrcorner \phi = 0\}^\perp \subset W$, where $\alpha \lrcorner \phi = \partial\phi/\partial\alpha$ is the partial derivative of ϕ by α. Then $\dim\langle\phi\rangle$ is the minimal number of variables needed to express ϕ in some coordinate system and $\phi \in S^d\langle\phi\rangle$. If $\langle\phi\rangle \neq W$ then the vanishing set $\mathrm{Zeros}(\phi) \subset \mathbb{P}W^*$ is a cone over $\{[\alpha] \mid \alpha \lrcorner \phi = 0\}$.

For $\phi \in S^dW$,

$$R_{v_d(\mathbb{P}W)}(\phi) = R_{v_d(\mathbb{P}\langle\phi\rangle)}(\phi), \tag{3}$$

$$\underline{R}_{v_d(\mathbb{P}W)}(\phi) = \underline{R}_{v_d(\mathbb{P}\langle\phi\rangle)}(\phi), \tag{4}$$

see, e.g., [12] or [14], Prop. 3.1.

Define the *subspace variety*

$$\mathrm{Sub}_k = \mathbb{P}\{\phi \in S^dW \mid \dim\langle\phi\rangle \leq k\}.$$

Defining equations of Sub_k are given by the $(k+1)\times(k+1)$ minors of $\phi_{1,d-1}$ (see, e.g, [19] §7.2), so in particular

$$\sigma_k(v_d(\mathbb{P}W)) \subseteq \mathrm{Sub}_k, \tag{5}$$

i.e., $[\phi] \in \sigma_r(v_d(\mathbb{P}W))$ implies $\dim\langle\phi\rangle \leq r$. We will often work by induction and assume $\langle\phi\rangle = W$. In particular, we often restrict attention to $\sigma_r(v_d(\mathbb{P}W))$ for $r \geq \dim W$.

3.3. **Specialization.** If $X \subset \mathbb{P}V$ is a variety and we consider the image of the cone $\hat{X} \subset V$ under a projection $\pi_U : V \to (V/U)$ where $U \subset V$ is a subspace, then for $p \in \mathbb{P}V$, $R_{\pi_U(X)}(\pi_U(p)) \leq R_X(p)$ and similarly for border rank. To see this, if $p = q_1 + \cdots + q_r$, then $\pi_U(p) = \pi_U(q_1) + \cdots + \pi_U(q_r)$ because π_U is a linear map. In particular, given a polynomial in $n+m$ variables, $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, if we set the y_i to be linear combinations of the x_j, then

$$R_{v_d(\mathbb{P}\mathbb{C}^n)}(\phi(x, y(x))) \leq R_{v_d(\mathbb{P}\mathbb{C}^{n+m})}(\phi(x, y)). \tag{6}$$

3.4. **Symmetric Flattenings (Catalecticant minors).** For $r < \frac{1}{n}\binom{n+d-1}{n-1}$, some equations for $\sigma_r(v_d(\mathbb{P}W))$ are known, but not enough to generate the ideal in most cases. The main known equations come from *symmetric flattenings*, also known as *catalecticant matrices*, and date back to Macaulay [15]. Other equations are discussed in [16], and there is recent work describing general methods for obtaining further equations, see [13]. Here are the symmetric flattenings:

For $\phi \in S^dW$, define the contracted maps

$$\phi_{s,d-s} : S^sW^* \times S^{d-s}W^* \to \mathbb{C}. \tag{7}$$

Then we may consider the left and right kernels $\mathrm{Lker}\,\phi_{s,d-s} \subseteq S^sW^*$, $\mathrm{Rker}\,\phi_{s,d-s} \subseteq S^{d-s}W^*$. We will abuse notation and identify $\phi_{s,d-s}$ with the associated map $S^{d-s}W^* \to S^sW$. We restrict attention to $\phi_{s,d-s}$ for $1 \leq s \leq \lfloor d/2 \rfloor$ to avoid redundancies.

Remark 3.1. The sequence $\{\operatorname{rank}\phi_{s,d-s} : 1 \le s \le \lfloor \frac{d}{2} \rfloor\}$ may decrease, as observed by Stanley [18, Example 4.3]. For instance, let

$$\phi = x_1x_{11}^3 + x_2x_{11}^2x_{12} + x_3x_{11}^2x_{13} + x_4x_{11}x_{12}^2 + x_5x_{11}x_{12}x_{13} + x_6x_{11}x_{13}^2 + x_7x_{12}^3 + x_8x_{12}^2x_{13} + x_9x_{12}x_{13}^2 + x_{10}x_{13}^3.$$

Then $\operatorname{rank}\phi_{1,3} = 13$ but $\operatorname{rank}\phi_{2,2} = 12$. On the other hand, Stanley showed that if $\dim W \le 3$ and $\phi \in S^dW$, then $\operatorname{rank}\phi_{s,d-s}$ is nondecreasing in $1 \le s \le \lfloor \frac{d}{2} \rfloor$ [18, Theorem 4.2].

Remark 3.2. When $\dim W = 2$, for any value of s such that $s, d-s \ge r+1$, one obtains a set of generators for $I(\sigma_r(v_d(\mathbb{P}W))$ by the $r+1$ by $r+1$ minors of the $s, d-s$ symmetric flattening, see [10]. Also when $r = 2$, taking $s = 1$ and $s = 2$ is enough to obtain generators of $I(\sigma_2(v_d(\mathbb{P}W)))$, see [11]. The hypersurface $\sigma_3(v_3(\mathbb{P}^2))$ is given by a degree four equation called the *Aronhold invariant*, which does not arise as a symmetric flattening (see [16]). Very few other cases are understood.

3.5. **A classical lower bound for rank.** The following is a symmetric analog of a result that is well known for tensors, e.g. [3] Proposition 14.45.

Proposition 3.3. *$R(\phi)$ is at least the minimal number of elements of $v_s(\mathbb{P}W)$ needed to span (a space containing) $\mathbb{P}(\phi_{s,d-s}(S^{d-s}W^*))$.*

Proof. If $\phi = \eta_1^d + \cdots + \eta_r^d$, then $\phi_{s,d-s}(S^{d-s}W^*) \subseteq \langle \eta_1^s, \ldots, \eta_r^s \rangle$. □

3.6. **Spaces of polynomials where the possible ranks and border ranks are known.** The only cases are as follows. (i.) $S^2\mathbb{C}^n$ for all n. Here rank and border rank coincide with the rank of the corresponding symmetric matrix, and there is a normal form for elements of rank r, namely $x_1^2 + \cdots + x_r^2$. (ii.) $S^d\mathbb{C}^2$ where the possible ranks and border ranks are known, see Theorem 4.1. However there are no normal forms in general. (iii.) $S^3\mathbb{C}^3$ where the possible ranks and border ranks were determined in [5]. We also explicitly describe which normal forms have which ranks in §5. The normal forms date back to [20].

4. The theorem of Comas and Seiguer

Theorem 4.1 (Comas-Seiguer, [4])**.** *Consider $v_d(\mathbb{P}^1) \subset \mathbb{P}^d$, and recall that $\sigma_{\lfloor \frac{d+1}{2} \rfloor}(v_d(\mathbb{P}^1)) = \mathbb{P}^d$. Let $r \le \lfloor \frac{d+1}{2} \rfloor$. Then*

$$\sigma_r(v_d(\mathbb{P}^1)) = \{[\phi] : R(\phi) \le r\} \cup \{[\phi] : R(\phi) \ge d - r + 2\}.$$

Equivalently, for all $r > 1$,

$$\sigma_r(v_d(\mathbb{P}^1)) \setminus \sigma_{r-1}(v_d(\mathbb{P}^1)) = \{[\phi] : R(\phi) = r\} \cup \{[\phi] : R(\phi) = d - r + 2\}.$$

Throughout this section we write $W = \mathbb{C}^2$.

Lemma 4.2. *Let $\phi \in S^d(W)$. Let $1 \le r \le d-1$. Then $R(\phi) > r$ if and only if $\mathbb{P}\operatorname{Lker}\phi_{r,d-r} \subset v_r(\mathbb{P}W)^\vee$.*

Proof. First say$R(\phi) \le r$ and write $\phi = w_1^d + \cdots + w_r^d$. Then $\operatorname{Lker}\phi_{r,d-r}$ contains the polynomial with distinct roots $w_1, \ldots, w_r$. Conversely, say $0 \ne P \in \operatorname{Lker}\phi_{r,d-r}$ has distinct roots $w_1, \ldots, w_r$. It will be sufficient to show $\phi \wedge w_1^d \wedge \ldots \wedge w_r^d = 0$. We show $\phi \wedge w_1^d \wedge \ldots \wedge w_r^d(p_1, \ldots, p_{r+1}) = 0$ for all $p_1, \ldots, p_{r+1} \in S^dW^*$ to finish the proof. Rewrite this as

$$\phi(p_1)m_1 - \phi(p_2)m_2 + \cdots + (-1)^r\phi(p_{r+1})m_{r+1} = \phi(m_1p_1 + \cdots + (-1)^r m_{r+1}p_{r+1})$$

where $m_j = w_1^d \wedge \cdots \wedge w_r^d(p_1, \ldots, \hat{p}_j, \ldots, p_{r+1}) \in \mathbb{C}$ (considering $S^d W$ as the dual vector space to $S^d W^*$). Now for each j,

$$\begin{aligned} w_j^d(m_1 p_1 + \cdots + (-1)^r m_{r+1} p_{r+1}) &= \sum_{i=1}^{r+1} w_j^d((-1)^{i-1} m_i p_i) \\ &= \sum_{i=1}^{r+1} (-1)^{2(i-1)} w_j^d \wedge w_1^d \wedge \cdots \wedge w_r^d(p_1, \ldots, p_{r+1}) \\ &= 0. \end{aligned}$$

Hence, now considering the p_j as polynomials of degree d on W,

$$(m_1 p_1 + \cdots + (-1)^r m_{r+1} p_{r+1})(w_i) = 0$$

for each i. But then $(m_1 p_1 + \cdots + (-1)^r m_{r+1} p_{r+1}) = PQ$ for some $Q \in S^{d-r} W^*$ and $\phi(PQ) = 0$ because $P \in \operatorname{Lker} \phi_{r,d-r}$. □

As mentioned above, the generators of the ideal of $\sigma_r(v_d(\mathbb{P}^1))$ can be obtained from the $(r+1) \times (r+1)$ minors of $\phi_{s,d-s}$. Thus (see [6] for more details):

Lemma 4.3. *For $\phi \in S^d \mathbb{C}^2$ and $1 \le r \le \lfloor d/2 \rfloor$ the following are equivalent.*

(1) $[\phi] \in \sigma_r(v_d(\mathbb{P}^1))$,
(2) $\operatorname{rank} \phi_{s,d-s} \le r$ *for* $s = \lfloor d/2 \rfloor$,
(3) $\operatorname{rank} \phi_{r,d-r} \le r$,
(4) $\operatorname{Lker} \phi_{r,d-r} \neq \{0\}$.

Lemma 4.4. *Let $r \le \lfloor \frac{d+1}{2} \rfloor$. If $\phi = \eta_1^d + \cdots + \eta_k^d$, $k \le d - r + 1$, and $P \in \operatorname{Lker} \phi_{r,d-r}$, then $P(\eta_i) = 0$ for each $1 \le i \le k$.*

Proof. For $1 \le i \le k$ let $M_i \in W^*$ annihilate η_i. In particular, $M_i(\eta_j) \neq 0$ if $j \neq i$, because the $[\eta_j]$ are distinct: η_j is not a multiple of η_i. Let $L \in W^*$ not vanish at any η_i. For each i, let

$$g_i = P M_1 \cdots \widehat{M_i} \cdots M_k L^{d-r+1-k},$$

so $\deg g_i = d$. Since $P \in \operatorname{Lker} \phi_{r,d-r}$ we get $\phi(g_i) = 0$. On the other hand, $\eta_j^d(g_i) = 0$ for $j \neq i$, so

$$\eta_i^d(g_i) = 0 = P(\eta_i) M_1(\eta_i) \cdots \widehat{M_i(\eta_i)} \cdots M_k(\eta_i) L(\eta_i)^{d-r+1-k}.$$

All the factors on the right are nonzero except possibly $P(\eta_i)$. Thus $P(\eta_i) = 0$. □

Proof of Theorem 4.1. Suppose $[\phi] \in \sigma_r(v_d(\mathbb{P}^1))$ and $R(\phi) \le d - r + 1$. Write $\phi = \eta_1^d + \cdots + \eta_k^d$ for some $k \le d - r + 1$ and the $[\eta_i]$ distinct. $[\phi] \in \sigma_r(v_d(\mathbb{P}^1))$ implies $\operatorname{rank} \phi_{r,d-r} \le r$, so $\dim \operatorname{Lker} \phi_{r,d-r} \ge 1$. Therefore there is some nonzero $P \in \operatorname{Lker} \phi_{r,d-r}$. Every $[\eta_i]$ is a zero of P, but $\deg P = r$ so P has at most r roots. So in fact $k \le r$. This shows the inclusion $\subseteq$ in the statement of the theorem.

To show $\{[\phi] : R(\phi) \ge d-r+2\} \subseteq \sigma_r(v_d(\mathbb{P}^1))$, suppose $R(\phi) \ge d-r+2$ and $[\phi] \notin \sigma_{r-1}(v_d(\mathbb{P}^1))$. Then $\operatorname{codim} \operatorname{Rker} \phi_{r-1,d-r+1} = r$ by Lemma 4.3, and $\mathbb{P} \operatorname{Rker} \phi_{r-1,d-r+1} \subset v_r(\mathbb{P}W)^\vee$ by Lemma 4.2 (applied to $\operatorname{Rker} \phi_{r-1,d-r+1} = \operatorname{Lker} \phi_{d-r+1,r-1}$). This means every polynomial $P \in \operatorname{Rker} \phi_{r-1,d-r+1}$ has a singularity (multiple root in $\mathbb{P}^1$). By Bertini's theorem, there is a basepoint of the linear system (a common divisor of all the polynomials in $\operatorname{Rker} \phi_{r-1,d-r+1}$). Let F be the greatest common divisor. Say $\deg F = f$. Let $M = \{P/F \mid P \in \operatorname{Rker} \phi_{r-1,d-r+1}\}$. Every $P/F \in M$ has degree $d - r + 1 - f$. So $\mathbb{P}M \subset \mathbb{P}S^{d-r+1-f} W^*$, which has dimension $d - r + 1 - f$. Also $\dim \mathbb{P}M = \dim \mathbb{P} \operatorname{Rker} \phi_{r-1,d-r+1} = d - 2r + 1$. Therefore $d - 2r + 1 \le d - r + 1 - f$, so $f \le r$.

Since the polynomials in M have no common roots, $(S^{r-f}W^*).M = S^{d-2f+1}W^*$ (see, e.g. [9], Lemma 9.8). Thus

$$S^{r-1}W^*.\operatorname{Rker}\phi_{r-1,d-r+1} = S^{f-1}W^*.S^{r-f}W^*.M.F = S^{d-f}W^*.F.$$

So if $Q \in S^{d-f}W^*$, then $FQ = GP$ for some $G \in S^{r-1}W^*$ and $P \in \operatorname{Rker}\phi_{r-1,d-r+1}$, so $\phi(FQ) = \phi(GP) = 0$. Thus $0 \neq F \in \operatorname{Lker}\phi_{f,d-f}$, so $[\phi] \in \sigma_f(v_d(\mathbb{P}^1))$. And finally $\sigma_f(v_d(\mathbb{P}^1)) \subset \sigma_r(v_d(\mathbb{P}^1))$, since $f \leq r$. $\square$

Corollary 4.5. *If $a, b > 0$ then $R(x^a y^b) = \max(a+1, b+1)$.*

Proof. Assume $a \leq b$. The symmetric flattening $(x^a y^b)_{a,b}$ has rank $a+1$ (the image is spanned by $x^a y^0, x^{a-1}y^1, \ldots, x^0 y^a$); it follows that $\underline{R}(x^a y^b) \geq a+1$. Similarly, $(x^a y^b)_{a+1,b-1}$ has rank $a+1$ as well. Therefore $\underline{R}(x^a y^b) = a+1$, so $R(x^a y^b)$ is either $b+1$ or $a+1$.

Let $\{\alpha, \beta\}$ be a dual basis to $\{x, y\}$. If $a < b$ then $\mathbb{P}\operatorname{Lker}(x^a y^b)_{a+1,b-1} = \{[\alpha^{a+1}]\} \subset v_{a+1}(\mathbb{P}W)^\vee$. Therefore $R(x^a y^b) > a+1$. If $a = b$ then $R(x^a y^b) = a+1 = b+1$. $\square$

In particular, $R(x^{n-1}y) = n$.

5. Maximum rank of arbitrary varieties

For any variety $X \subset \mathbb{P}V = \mathbb{P}^N$ that is not contained in a hyperplane, *a priori* the maximum X-rank of any point is $N+1$ as we may take a basis of V consisting of elements of X. This maximum occurs if, e.g., X is a collection of $N+1$ points.

Proposition 5.1. *Let $X \subset \mathbb{P}^N = \mathbb{P}V$ be a smooth irreducible variety of dimension n not contained in a hyperplane. Then for all $p \in \mathbb{P}V$, $R_X(p) \leq N+1-n$.*

Proof. If $p \in X$ then $R_X(p) = 1 \leq N+1-n$. Henceforth we consider only $p \notin X$.

If L is an $(N-n)$-dimensional linear subspace in $\mathbb{P}V$ intersecting X transversely, then, since X spans $\mathbb{P}V$ and X is irreducible, $L \cap X$ spans L (see [7], pg. 174). If p lies in any such L, we may choose a basis for L consisting of $\dim L + 1 = N+1-n$ distinct points from $L \cap X$, so $R_X(p) \leq N+1-n$.

We proceed by induction on the dimension of X. If $\dim X = 1$, consider the set of hyperplanes $\mathcal{H}_p := \{M \in \mathbb{P}V^* \mid p \in M\}$. By Bertini's theorem, for a general element $M \in \mathcal{H}_p$, $M \cap X$ is nonsingular outside $\operatorname{base}(\mathcal{H}_p) := \bigcap_{M \in \mathcal{H}_p}(M \cap X)$, the base locus of the linear system. But $\bigcap_{M \in \mathcal{H}_p} M = \{p\} \not\subset X$, so $\operatorname{base}(\mathcal{H}_p)$ is empty. So for a general hyperplane M through p, $M \cap X$ is nonsingular. Since $M \cap X$ is zero-dimensional, M must intersect X transversely. Therefore $R_X(p) \leq N+1-n = N$.

For the inductive step, define $\mathcal{H}_p$ as above. Again for general $M \in \mathcal{H}_p$, $M \cap X$ is nonsingular by Bertini's theorem, spans M, and is also irreducible if $\dim X = n > 1$. (The last assertion follows from the Lefschetz hyperplane theorem, which implies that when $\dim X > 1$, the restriction $H^0(X) \to H^0(M \cap X)$ is surjective, so every $M \cap X$ is connected. If $M \cap X$ is reducible, then by the connectedness the components of $M \cap X$ must meet, and the points of intersection of components are necessarily singular. But by Bertini's theorem a general $M \cap X$ is smooth, hence irreducible.) Note that $\dim M \cap X = n-1$ and $\dim M = N-1$. Thus by induction, $R_{M\cap X}(p) \leq (N-1)+1-(n-1) = N+1-n$. Since $M \cap X \subset X$ we have $R_X(p) \leq R_{M\cap X}(p) \leq N+1-n$. $\square$

In particular:

Corollary 5.2. *Given $\phi \in S^d\mathbb{C}^n$, $R(\phi) \leq \binom{n+d-1}{d} + 1 - n$.*

Corollary 5.3. *Let $C \subset \mathbb{P}^N = \mathbb{P}V$ be a smooth curve not contained in a hyperplane. Then the maximum C-rank of any $p \in \mathbb{P}V$ is at most N.*

For $X \subset \mathbb{P}^N$ a rational normal curve of degree N, the maximal X-rank is indeed N (see §4). Since rational normal curves are curves of minimal degree, we ask:

Question 5.4. If $C \subset \mathbb{P}^N$ is an irreducible curve not lying in any hyperplane and there is a point $p \in \mathbb{P}^N$ with $R_C(p) = N$, must C be a rational normal curve?

We may refine the above discussion to ask, what is the maximum X-rank of a point lying on a given secant variety of X, that is, with a bounded X-border rank? For any X, essentially by definition, $\{x \in \sigma_2(X) \mid R_X(x) > 2\} \subseteq \tau(X)\backslash X$. The rank of a point on $\tau(X)$ can already be the maximum, as well as being arbitrarily large. Both these occur for X a rational normal curve of degree d (see §4) where the rank of a point on $\tau(X)$ is the maximum d.

6. Proof and variants of Theorem 1.3

For $\phi \in S^d W$ and $s \geq 0$, let

$$\Sigma_s(\phi) = \Sigma_s := \{[\alpha] \in \mathrm{Zeros}(\phi) \mid \mathrm{mult}_{[\alpha]}(\phi) \geq s+1\} \subset \mathbb{P}W^*.$$

This definition agrees with our coordinate definition in §1.

Remark 6.1. Note that for $\phi \in S^d W$, $\Sigma_d = \emptyset$ and $\Sigma_{d-1} = \mathbb{P}\langle\phi\rangle^\perp$. In particular, Σ_{d-1} is empty if and only if $\langle\phi\rangle = W$.

Remark 6.2. The stratification mentioned in the introduction is identified as

$$v_d(\mathbb{P}W^*)_k{}^\vee = \mathbb{P}\{\phi \mid \Sigma_{k-1}(\phi) \neq \emptyset\}.$$

It is natural to refine this stratification by the geometry of Σ_{k-1}, for example by:

$$v_d(\mathbb{P}W^*)_{k,a}{}^\vee := \mathbb{P}\{\phi \mid \dim \Sigma_{k-1}(\phi) \geq a\}.$$

Proposition 6.3.

$$v_{d-s}(\Sigma_s) = \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} \cap v_{d-s}(\mathbb{P}W^*).$$

That is, $[\alpha] \in \Sigma_s$ if and only if $[\alpha^{d-s}] \in \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}$.

Proof. For all $\alpha \in W^*$ and $w_1, \dots, w_s \in W^*$,

$$\tilde{\phi}(w_1, \dots, w_s, \alpha, \dots, \alpha) = \left(\frac{\partial^s \phi}{\partial w_1 \cdots \partial w_s}\right)(\alpha).$$

Now $\alpha^{d-s} \in \mathrm{Rker}\,\phi_{s,d-s}$ if and only if the left hand side vanishes for all $w_1, \dots, w_s$, and $\mathrm{mult}_{[\alpha]}\phi \geq s+1$ if and only if the right hand side vanishes for all $w_1, \dots, w_s$. □

Lemma 6.4. *Let $\phi \in S^d W$. Suppose we have an expression $\phi = \eta_1^d + \cdots + \eta_r^d$. Let $L := \mathbb{P}\{p \in S^{d-s}W^* \mid p(\eta_i) = 0, 1 \leq i \leq r\}$. Then*

1. *$L \subset \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}$.*
2. *$\mathrm{codim}\, L \leq r$.*
3. *If $\langle\phi\rangle = W$, then $L \cap v_{d-s}(\mathbb{P}W^*) = \emptyset$.*

Proof. For the first statement, for $p \in S^{d-s}W^*$ and any $q \in S^s W^*$,

$$\phi_{s,d-s}(q)(p) = q(\eta_1)p(\eta_1) + \cdots + q(\eta_r)p(\eta_r).$$

If $[p] \in L$ then each $p(\eta_i) = 0$, so $\phi_{s,d-s}(q)(p) = 0$ for all q. Therefore $p \in \mathrm{Rker}\,\phi_{s,d-s}$.

The second statement is well-known. Since each point $[\eta_i]$ imposes a single linear condition on the coefficients of p, L is the common zero locus of a system of r linear equations. Therefore $\mathrm{codim}\, L \leq r$.

If $\langle\phi\rangle = W$, then $W = \langle\phi\rangle \subseteq \langle\eta_1, \dots, \eta_r\rangle \subseteq W$, so the η_i span W. Therefore the points $[\eta_i]$ in $\mathbb{P}W$ do not lie on any hyperplane. If $L \cap v_{d-s}(\mathbb{P}W^*) \neq \emptyset$, say $[\alpha^{d-s}] \in L$, then the linear form α vanishes at each $[\eta_i]$, so the $[\eta_i]$ lie on the hyperplane defined by α, a contradiction. □

Proof of Theorem 1.3. Suppose $\phi = \eta_1^d + \cdots + \eta_r^d$. Consider the linear series $L = \mathbb{P}\{p \in S^{d-s}W^* \mid p(\eta_i) = 0, 1 \leq i \leq r\}$ as in Lemma 6.4. Then L is contained in $\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}$ so

$$r \geq \mathrm{codim}\, L \geq \mathrm{codim}\, \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} = \mathrm{rank}\,\phi_{s,d-s}.$$

Remark 6.5. Note that taking $r = R(\phi)$ proves equation (1), *a priori* just dealing with rank, but in fact also for border rank by the definition of Zariski closure.

Now since $\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}$ is a projective space, if $\dim L + \dim \Sigma_s \geq \dim \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}$ we would have $L \cap \left(v_{d-s}(\mathbb{P}W^*) \cap \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}\right) \neq \emptyset$. But by Lemma 6.4 this intersection is empty. Therefore

$$\dim L + \dim \Sigma_s < \dim \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}.$$

Taking codimensions in $\mathbb{P}S^{d-s}W^*$, we may rewrite this as

$$\mathrm{codim}\, L - \dim \Sigma_s > \mathrm{codim}\, \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} = \mathrm{rank}\,\phi_{s,d-s}.$$

Taking $r = R(\phi)$ yields $R(\phi) \geq \mathrm{codim}\, L > \mathrm{rank}\,\phi_{s,d-s} + \dim \Sigma_s$. □

Remark 6.6. If $\phi \in S^dW$ with $\langle \phi \rangle = W$ and $R(\phi) = n = \dim W$, then the above theorem implies $\Sigma_1 = \emptyset$. Note that this is easy to see directly: Writing $\phi = \eta_1^d + \cdots + \eta_n^d$, we must have $\langle \eta_1, \ldots, \eta_n \rangle = \langle \phi \rangle = W$, so in fact the η_i are a basis for W. Then the singular set of $\mathrm{Zeros}(\phi)$ is the common zero locus of the derivatives η_i^{d-1} in $\mathbb{P}W$, which is empty.

Remark 6.7. The assumption that $\langle \phi \rangle = W$ is equivalent to $\mathrm{Lker}\,\phi_{1,d-1} = \{0\}$, i.e., that $\mathrm{Zeros}(\phi)$ is not a cone over a variety in a lower-dimension subspace. It would be interesting to have a geometric characterization of the condition $\mathrm{Lker}\,\phi_{k,d-k} = \{0\}$ for $k > 1$.

Question 6.8. Can Theorem 1.3 be strengthened in the case Σ_s has several components, e.g., when it is a collection of points?

In the following sections we apply Theorem 1.3 to several classes of polynomials. Before proceeding we note the following extension.

Proposition 6.9. *If* $\mathrm{Lker}\,\phi_{2,d-2} = \{0\}$ *then for each* s*,*

$$R(\phi) \geq \mathrm{rank}\,\phi_{s,d-s} + \dim \left[\bigcup_{\substack{\beta \in W^* \\ \beta \neq 0}} \Sigma_s(\partial\phi/\partial\beta) \right] + 1.$$

In particular, for every $\beta \neq 0$, $\Sigma_{s+1}(\phi) \subseteq \Sigma_s(\partial\phi/\partial\beta)$.

Proof. Any point on the tangential variety to the Veronese $\tau(v_{d-s}(\mathbb{P}W^*))$ has the form $[\alpha^{d-s-1}\beta]$ for some $[\alpha], [\beta] \in \mathbb{P}W^*$ (possibly $[\alpha] = [\beta]$). We have

$$[\alpha^{d-s-1}\beta] \in \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} \cap \tau(v_{d-s}(\mathbb{P}W^*))$$

if and only if $[\alpha^{d-s-1}] \in \mathbb{P}\,\mathrm{Rker}(\partial\phi/\partial\beta)_{s,d-s-1}$. By Proposition 6.3, this occurs if and only if $\partial\phi/\partial\beta$ vanishes to order $> s$ at α. This proves

$$\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} \cap \tau(v_{d-s}(\mathbb{P}W^*)) = \bigcup_{\beta \in W^*} \Sigma_s(\partial\phi/\partial\beta).$$

Now, suppose $\phi = \eta_1^d + \cdots + \eta_r^d$ and let $L \subset \mathbb{P}S^{d-s}(W^*)$ be the set of hypersurfaces of degree $d-s$ containing each $[\eta_i]$. As before, L is a linear subspace contained in $\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}$. Suppose $[\alpha^{d-s-1}\beta] \in L$ for some $\alpha, \beta \in W^*$. That is, each $[\eta_i]$ is annihilated by the polynomial $\alpha^{d-s-1}\beta$. Hence each $[\eta_i]$ is in fact annihilated by the polynomial $\alpha\beta$. Therefore $[\alpha\beta] \in \mathbb{P}\,\mathrm{Lker}\,\phi_{2,d-2}$.

The assumption that $\mathrm{Lker}\,\phi_{2,d-2} = \{0\}$ thus implies L is disjoint from $\tau(v_{d-s}(\mathbb{P}W^*))$.

Description	normal form	R	$\underline{R}$	$\mathbb{P}\,\mathrm{Rker}\,\phi_{1,2} \cap \sigma_2(v_2(\mathbb{P}W^*))$
triple line	x^3	1	1	
three concurrent lines	$xy(x+y)$	2	2	
double line + line	x^2y	3	2	
irreducible	$y^2z - x^3 - z^3$	3	3	triangle
irreducible	$y^2z - x^3 - xz^2$	4	4	smooth
cusp	$y^2z - x^3$	4	3	double line + line
triangle	xyz	4	4	triangle
conic + transversal line	$x(x^2+yz)$	4	4	conic + transversal line
irreducible	$y^2z - x^3 - axz^2 - bz^3$	4	4	irred. cubic, smooth for general a, b
conic + tangent line	$y(x^2+yz)$	5	3	triple line

TABLE 1. Ranks and border ranks of plane cubic curves.

In the projective space $\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s}$,

$$L \cap (\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} \cap \tau(v_{d-s}(\mathbb{P}W^*))) = \emptyset.$$

As in the previous theorem, we count dimensions. Two varieties in a projective space can be disjoint only if their dimensions sum to less than the ambient dimension, so

$$\dim L + \dim(\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} \cap \tau(v_{d-s}(\mathbb{P}W^*))) < \dim \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s},$$

and taking codimensions in $\mathbb{P}S^{d-s}W^*$ yields the inequality

$$r \geq \operatorname{codim} L > \operatorname{codim} \mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} + \dim(\mathbb{P}\,\mathrm{Rker}\,\phi_{s,d-s} \cap \tau(v_{d-s}(\mathbb{P}W^*)))\,.$$

Finally take $r = R(\phi)$ and identify the intersection on the right hand side with the union of Σ_s's of derivatives of ϕ as above. $\square$

One step in the proof above generalizes slightly: With L as in the proof, if $[D] \in \mathbb{P}L$ and D factors as $D = \alpha_1^{a_1} \cdots \alpha_k^{a_k}$, then $\alpha_1 \cdots \alpha_k \in \mathrm{Lker}\,\phi_{k,d-k}$. In fact, this idea already appeared in the proof of Theorem 1.3 in the case $D = \alpha^s$.

7. Plane cubic curves

Normal forms for plane cubic curves were determined in [20] in the 1930's. In [5] an explicit algorithm was given for determining the rank of a cubic curve (building on unpublished work of Reznick), and determined the possible ranks for polynomials in each $\sigma_r(v_3(\mathbb{P}^2))\backslash\sigma_{r-1}(v_3(\mathbb{P}^2))$. Here we give the explicit list of normal forms and their ranks and border ranks, illustrating how one can use singularities of auxilliary geometric objects to determine the rank of a polynomial.

Theorem 7.1. *The possible ranks and border ranks of plane cubic curves are described in Table 1.*

The proof of Theorem 7.1 given by [5] relies first on a computation of equations for the secant varieties $\sigma_k(v_3(\mathbb{P}W))$, $\dim W = 3$, for $2 \leq k \leq 3$, which determines all the border ranks in Table 1. Note that $\sigma_3(v_3(\mathbb{P}W))$ is a hypersurface defined by the Aronhold invariant, not a symmetric flattening. To refine the results to give the ranks of a non-generic point ϕ in each secant variety, first [5] uses the geometry of the *Hessian* of ϕ to distinguish some cases. (The Hessian is the variety whose equation is the determinant of the Hessian matrix of the equation of ϕ. Given a vector $v \in W^*$, $\phi_{1,2}(v)$ in bases is the Hessian of ϕ evaluated at v.)

The last case, $\phi = y(x^2+yz)$, is distinguished by an unpublished argument due to B. Reznick. Reznick shows by direct calculation that for any linear form L, the geometry of the Hessian of

$\phi - L^3$ implies $\phi - L^3$ has rank strictly greater than 3; so ϕ itself has rank strictly greater than 4. We thank Reznick for sharing the details of this argument with us.

When the curve $\mathrm{Zeros}(\phi) \subset \mathbb{P}W^*$ is not a cone, the variety $\mathbb{P}\,\mathrm{Rker}\,\phi_{1,2} \cap \sigma_2(v_2(\mathbb{P}W^*))$ is the Hessian cubic of ϕ.

We exploit this connection to prove Theorem 7.1 by examining the geometry of the Hessian using the machinery we have set up to study $\mathbb{P}\,\mathrm{Rker}\,\phi_{1,2}$. We begin by computing the ranks of each cubic form. We show that $\phi = y(x^2 + yz)$ has rank 5 by directly studying the Hessian of ϕ itself (rather than the modification $\phi - L^3$ as was done by Reznick).

Proof. Upper bounds for the ranks listed in Table 1 are given by simply displaying an expression involving the appropriate number of terms. For example, to show $R(xyz) \leq 4$, observe that

$$xyz = \frac{1}{24}\Big((x+y+z)^3 - (-x+y+z)^3 - (x-y+z)^3 - (x+y-z)^3\Big).$$

We present the remainder of these expressions in Table 3.

Next we show lower bounds for the ranks listed in Table 1. The first three cases are covered by Theorem 4.1. For all the remaining ϕ in the table, $\phi \notin \mathrm{Sub}_2$, so by (5), $R(\phi) \geq 3$. By Remark 6.6, if ϕ is singular then $R(\phi) \geq 4$, and this is the case for the cusp, the triangle, and the union of a conic and a line. We have settled all but the following three cases:

$$y^2z - x^3 - xz^2, \quad y^2z - x^3 - axz^2 - bz^3, \quad y(x^2 + yz).$$

If $\phi = \eta_1^3 + \eta_2^3 + \eta_3^3$ with $[\eta_i]$ linearly independent, then the Hessian of ϕ is defined by $\eta_1\eta_2\eta_3 = 0$, so it is a union of three nonconcurrent lines. In particular, it has three distinct singular points. But a short calculation verifies that the Hessian of $y^2z - x^3 - xz^2$ is smooth and the Hessian of $y^2z - x^3 - axz^2 - bz^3$ has at most one singularity. Therefore these two curves have rank at least 4, which agrees with the upper bounds given in Table 3.

Let $\phi = y(x^2 + yz)$. The Hessian of ϕ is defined by the equation $y^3 = 0$. Therefore the Hessian $\mathbb{P}\,\mathrm{Rker}\,\phi_{1,2} \cap \sigma_2(v_2(\mathbb{P}W))$ is a (triple) line. Since it is not a triangle, $R(y(x^2 + yz)) \geq 4$, as we have argued in the last two cases. But in this case we can say more.

Suppose $\phi = y(x^2 + yz) = \eta_1^3 + \eta_2^3 + \eta_3^3 + \eta_4^3$, with the $[\eta_i]$ distinct points in $\mathbb{P}W$. Since $\langle\phi\rangle = W$, the $[\eta_i]$ are not all collinear. Therefore there is a unique 2-dimensional linear space of quadratic forms vanishing at the η_i. These quadratic forms thus lie in $\mathrm{Rker}\,\phi_{1,2}$. In the plane $\mathbb{P}\,\mathrm{Rker}\,\phi_{1,2} \cong \mathbb{P}^2$, $H := \mathbb{P}\,\mathrm{Rker}\,\phi_{1,2} \cap \sigma_2(v_2(\mathbb{P}W))$ is a triple line and the pencil of quadratic forms vanishing at each η_i is also a line L.

Now either $H = L$ or $H \neq L$. If $H = L$, then L contains the point $\mathbb{P}\,\mathrm{Rker}\,\phi_{1,2} \cap v_2(\mathbb{P}W) \cong \Sigma_1$. But $\langle\phi\rangle = W$, so L is disjoint from $v_2(\mathbb{P}W)$. Therefore $H \neq L$. But then L contains exactly one reducible conic, corresponding to the point $H \cap L$. But this is impossible: a pencil of conics through four points in $\mathbb{P}^2$ contains at least three reducible conics (namely the pairs of lines through pairs of points).

Thus $\phi = y(x^2 + yz) = \eta_1^3 + \eta_2^3 + \eta_3^3 + \eta_4^3$ is impossible, so $R(y(x^2 + yz)) \geq 5$.

In conclusion, we have obtained for each cubic curve ϕ listed in Table 1 a lower bound $R(\phi) \geq m$ which agrees with the upper bound $R(\phi) \leq m$ as shown in Table 3. This completes the proof of the calculation of ranks.

Finally one may either refer to the well-known characterization of degenerations of cubic curves to find the border ranks; see for example [20] or simply evaluate the defining equations of the various secant varieties on the normal forms. ◻

8. Ranks and border ranks of some cubic polynomials

Proposition 8.1. *Consider $\phi = x_1y_1z_1 + \cdots + x_my_mz_m \in S^3W$, where $W = \mathbb{C}^{3m}$. Then $R(\phi) = 4m = \frac{4}{3}\dim W$ and $\underline{R}(\phi) = 3m = \dim W$.*

Proof. We have $\langle\phi\rangle = W$, so $\operatorname{rank}\phi_{1,2} = \dim W = 3m$, and Σ_1 contains the set $\{x_1 = y_1 = x_2 = y_2 = \cdots = x_m = y_m = 0\}$. Thus Σ_1 has dimension at least $m-1$. So $R(\phi) \geq 4m$ by Proposition 1.3. On the other hand, each $x_i y_i z_i$ has rank 4 by Theorem 7.1, so $R(\phi) \leq 4m$.

Since $\underline{R}(xyz) = 3$, we have $\underline{R}(\phi) \leq 3m$. On the other hand, one simply computes the matrix of $\phi_{1,2}$ and observes that it is a block matrix with rank at least $3m$. Therefore $\underline{R}(\phi) = 3m$. □

Proposition 8.2. *Let $\mathbb{C}^{m+1}$ with $m > 1$ have linear coordinates $x, y_1, \dots, y_m$. Then,*
(1) $R(x(y_1^2 + \cdots + y_m^2)) = 2m$.
(2) $R(x(y_1^2 + \cdots + y_m^2) + x^3) = 2m$.

Proof. Write $\phi = x(y_1^2 + \cdots + y_m^2) \in S^3W = S^3\mathbb{C}^{m+1}$. We have $\langle\phi\rangle = W$, so $\operatorname{rank}\phi_{1,2} = \dim W = m+1$, and $\Sigma_1 = \{x = y_1^2 + \cdots + y_m^2 = 0\} \cup \{y_1 = \cdots = y_m = 0\}$. Thus Σ_1 has dimension $m-2$. So $R(\phi) \geq 2m$ by Proposition 1.3.

Let $a_1, \dots, a_m$ be nonzero complex numbers with $\sum a_i = 0$. Write

$$\begin{aligned}\phi &= xy_1^2 + \cdots + xy_m^2 \\ &= (xy_1^2 - a_1x^3) + \cdots + (xy_m^2 - a_mx^3) \\ &= x(y_1 + a_1^{1/2}x)(y_1 - a_1^{1/2}x) + \cdots + x(y_m + a_m^{1/2}x)(y_m - a_m^{1/2}x).\end{aligned}$$

Since each $x(y_j - a_j^{1/2}x_j)(y_j + a_j^{1/2}x_j)$ defines a union of three concurrent lines in the plane with coordinates x_j, y_j, we see that $R(xy_j^2 - a_jx^3) = 2$. Thus ϕ is the sum of m terms which each have rank 2, so $R(\phi) \leq 2m$.

The second statement follows by the same argument (with $\sum a_i = -1$). □

We have the bounds

$$m + 1 = \operatorname{rank}\phi_{1,2} \leq \underline{R}(\phi) \leq R(\phi) = 2m.$$

It would be interesting to know the border rank of $x(y_1^2 + \cdots + y_m^2)$ and $x(y_1^2 + \cdots + y_m^2) + x^3$.

Remark 8.3. In particular, $x(y_12 + y_22 + y_32)$ has rank exactly 6, which is strictly greater than the generic rank 5 of cubic forms in four variables, answering a question of Geramita. (See the remark following Prop. 6.3 of [6])

More generally,

Proposition 8.4. *Let $n = \dim W$ and $\phi \in S^d(W)$ with $\langle\phi\rangle = W$. If ϕ is reducible, then $R(\phi) \geq 2n-2$. If ϕ has a repeated factor, then $R(\phi) \geq 2n-1$.*

Proof. We have $\operatorname{rank}\phi_{1,d-1} = \dim W = n$. If $\phi = \chi\psi$ factors, then $\Sigma_1(\phi)$ includes the intersection $\{\chi = \psi = 0\}$, which has codimension 2 in $\mathbb{P}W \cong \mathbb{P}^{n-1}$. Therefore $R(\phi) \geq n + n - 3 + 1 = 2n - 2$.

If ϕ has a repeated factor, say ϕ is divisible by ψ^2, then Σ_1 includes the hypersurface $\{\psi = 0\}$, which has codimension 1. So $R(\phi) \geq n + n - 2 + 1 = 2n - 1$. □

Proposition 8.5. *Let $\phi = x^2u + y^2v + xyz \in S^3W$, $\dim W = 5$. Then $\underline{R}(\phi) = 5$ and $8 \leq R(\phi) \leq 9$.*

Proof. The upper bound follows from the expression

$$\phi = (x+y+2^{1/3}z)^3 - (2^{2/3}x+z)^3 - (2^{2/3}y+z)^3 - x^2(-u-3x+3y-3\cdot 2^{1/3}z) - y^2(-v+3x-3y-3\cdot 2^{1/3}z),$$

where the last two terms have the form a^2b; recall that $R(a^2b) = 3$.

To obtain the lower bound, note that the map $\phi_{1,2}$ is surjective, so $\operatorname{codim}\operatorname{Rker}\phi_{1,2} = \dim W = 5$. In particular, $\underline{R}(\phi) \geq 5$. The singular set $\Sigma_1 = \{x = y = 0\} \cong \mathbb{P}^2$. Therefore $R(\phi) \geq 5 + 2 + 1 = 8$.

The upper bound for border rank follows by techniques explained in §10. Explicitly, define 5 curves in W as follows:

$$a(t) = x + t(u - z), \quad b(t) = y + t(v - z), \quad c(t) = (x + y) + tz, \quad d(t) = x + 2y, \quad e(t) = x + 3y,$$

and for $t \neq 0$ let $P(t) \subset \mathbb{P}S^3W$ be the $\mathbb{P}^4$ spanned by $a^3, \dots, e^3$, so $P(t) \subset \sigma_5(v_3(\mathbb{P}W))$. A straightforward calculation as in §10, §11 shows that (after scaling coordinates) $[\phi]$ lies in $\lim_{t\to 0} P(t)$. $\square$

9. Determinants and permanents

Let X be an $n \times n$ matrix whose entries $x_{i,j}$ are variables forming a basis for W. Let $\det_n = \det X$ and per_n be the permanent of X.

In [8], L. Gurvits applied the equations for flattenings (1) to the determinant and permanent polynomials to observe, for each $1 \leq a \leq n - 1$,

$$\mathrm{rank}(\det_n)_{a,n-a} = \mathrm{rank}(\mathrm{per}_n)_{a,n-a} = \binom{n}{a}^2,$$

giving lower bounds for border rank. (In [8] he is only concerned with rank but he only uses (1) for lower bounds.) Indeed, the image of $(\det_n)_{a,n-a}$ is spanned by the determinants of $a \times a$ submatrices of X, and the image of $(\mathrm{per}_n)_{a,n-a}$ is spanned by the permanents of $a \times a$ submatrices of X. These are independent and number $\binom{n}{a}^2$. In the same paper Gurvitz also gives upper bounds as follows.

$$R(\det_n) \leq 2^{n-1} n!, \qquad R(\mathrm{per}_n) \leq 4^{n-1}.$$

The first bound follows by writing $\det_n$ as a sum of $n!$ terms, each of the form $x_1 \cdots x_n$, and applying Proposition 11.6: $R(x_1 \cdots x_n) \leq 2^{n-1}$. For the second bound, a variant of the Ryser formula for the permanent (see [17]) allows one to write per_n as a sum of 2^{n-1} terms, each of the form $x_1 \cdots x_n$:

$$\mathrm{per}_n = 2^{-n+1} \sum_{\substack{\epsilon \in \{-1,1\}^n \\ \epsilon_1 = 1}} \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \epsilon_i \epsilon_j x_{i,j},$$

the outer sum taken over n-tuples $(\epsilon_1 = 1, \epsilon_2, \dots, \epsilon_n)$. Note that each term in the outer sum is a product of n independent linear forms and there are 2^{n-1} terms. Applying Proposition 11.6 again gives the upper bound for $R(\mathrm{per}_n)$.

Now, we apply Theorem 1.3 to improve the lower bounds for rank. The determinant $\det_n$ vanishes to order $a + 1$ on a matrix A if and only if every minor of A of size $n - a$ vanishes. Thus $\Sigma_a(\det_n)$ is the locus of matrices of rank at most $n - a - 1$. This locus has dimension $n^2 - 1 - (a + 1)^2$. Therefore, for each a,

$$R(\det_n) \geq \binom{n}{a}^2 + n^2 - (a + 1)^2.$$

The right hand side is maximized at $a = \lfloor n/2 \rfloor$.

A crude lower bound for $\dim \Sigma_a(\mathrm{per}_n)$ is obtained as follows. If a matrix A has $a+1$ columns identically zero, then each term in per_n vanishes to order $a + 1$, so per_n vanishes to order at least $a + 1$. Therefore $\Sigma_a(\mathrm{per}_n)$ contains the set of matrices with $a + 1$ zero columns, which is a finite union of projective linear spaces of dimension $n(n - a - 1) - 1$. Therefore, for each a,

$$R(\mathrm{per}_n) \geq \binom{n}{a}^2 + n(n - a - 1).$$

Again, the right hand side is maximized at $a = \lfloor n/2 \rfloor$.

n	2	3	4	5	6	7	8
Upper bound for $R(\det_n)$	4	24	192	1,920	23,040	322,560	5,160,960
Lower bound for $R(\det_n)$	4	14	43	116	420	1,258	4,939
Lower bound for $\underline{R}(\det_n)$	4	9	36	100	400	1,225	4,900
Upper bound for $R(\mathrm{per}_n)$	4	16	64	256	1,024	4,096	16,384
Lower bound for $R(\mathrm{per}_n)$	4	12	40	110	412	1,246	4,924
Lower bound for $\underline{R}(\mathrm{per}_n)$	4	9	36	100	400	1,225	4,900

TABLE 2. Bounds for determinants and permanents.

See Table 2 for values of the upper bound for rank and lower bound for border rank obtained by Gurvitz and the lower bound for rank given here.

10. Limits of secant planes for Veronese varieties

10.1. **Limits of secant planes for arbitrary projective varieties.** Let $X \subset \mathbb{P}V$ be a projective variety. Recall that $\sigma_r^0(X)$ denotes the set of points on $\sigma_r(X)$ that lie on a $\mathbb{P}^{r-1}$ spanned by r points on X. We work inductively, so we assume we know the nature of points on $\sigma_{r-1}(X)$ and study points on $\sigma_r(X)\backslash(\sigma_r^0(X) \cup \sigma_{r-1}(X))$.

It is convenient to study the limiting r-planes as points on the Grassmannian in its Plücker embedding, $G(r, V) \subset \mathbb{P}(\bigwedge^r V)$. I.e., we consider the curve of r planes as being represented by $[x_1(t) \wedge \cdots \wedge x_r(t)]$, where $x_j(t) \subset \hat{X}\backslash 0$ and examine the limiting plane as $t \to 0$. (There is a unique such plane as the Grassmannian is compact.)

Let $[p] \in \sigma_r(X)$. Then there exist curves $x_1(t), \dots, x_r(t) \subset \hat{X}$ with $p \in \lim_{t\to 0}\langle x_1(t), \dots, x_r(t)\rangle$. We are interested in the case when $\dim\langle x_1(0), \dots, x_r(0)\rangle < r$. (Here $\langle v_1, \dots, v_k\rangle$ denotes the linear span of the vectors $v_1, \dots, v_k$.) Use the notation $x_j = x_j(0)$. Assume for the moment that $x_1, \dots, x_{r-1}$ are linearly independent. Then we may write $x_r = c_1x_1 + \cdots + c_{r-1}x_{r-1}$ for some constants $c_1, \dots, c_{r-1}$. Write each curve $x_j(t) = x_j + tx_j' + t^2x_j'' + \cdots$ where derivatives are taken at $t = 0$.

Consider the Taylor series

$$\begin{aligned}
&x_1(t) \wedge \cdots \wedge x_r(t) = \\
&(x_1 + tx_1' + t^2x_1'' + \cdots) \wedge \cdots \wedge (x_{r-1} + tx_{r-1}' + t^2x_{r-1}'' + \cdots) \wedge (x_r + tx_r' + t^2x_r'' + \cdots) \\
&= t((-1)^r(c_1x_1' + \cdots c_{r-1}x_{r-1}' - x_r') \wedge x_1 \wedge \cdots \wedge x_{r-1}) + t^2(...) + \cdots
\end{aligned}$$

If the t coefficient is nonzero, then p lies in the the r plane $\langle x_1, \dots, x_{r-1}, (c_1x_1' + \cdots c_{r-1}x_{r-1}' - x_r')\rangle$.

If the t coefficient is zero, then $c_1x_1' + \cdots + c_{r-1}x_{r-1}' - x_r' = e_1x_1 + \cdots e_{r-1}x_{r-1}$ for some constants $e_1, \dots, e_{r-1}$. In this case we must examine the t^2 coefficient of the expansion. It is

$$\left(\sum_{k=1}^{r-1} e_kx_k' + \sum_{j=1}^{r-1} c_jx_j'' - x_r''\right) \wedge x_1 \wedge \cdots \wedge x_{r-1}.$$

One continues to higher order terms if this is zero.

For example, when $r = 3$, the t^2 term is

$$x_1' \wedge x_2' \wedge x_3 + x_1' \wedge x_2 \wedge x_3' + x_1 \wedge x_2' \wedge x_3' + x_1'' \wedge x_2 \wedge x_3 + x_1 \wedge x_2'' \wedge x_3 + x_1 \wedge x_2 \wedge x_3''. \tag{8}$$

10.2. **Limits for Veronese varieties.** As explained in §10.1, for any smooth variety $X \subset \mathbb{P}V$, a point on $\sigma_2(X)$ is either a point of X, a point on an honest secant line (i.e., a point of X-rank two) or a point on a tangent line of X. For a Veronese variety all nonzero tangent vectors are equivalent. They are all of the form $x^d + x^{d-1}y$ (or equivalently $x^{d-1}z$), in particular they lie on a subspace variety Sub_2 and thus have rank d by Theorem 4.1. In summary:

Proposition 10.1. *If $p \in \sigma_2(v_d(\mathbb{P}W))$ then $R(p) = 1$, 2 or d. In these cases p respectively has the normal forms $x^d, x^d + y^d, x^{d-1}y$. (The last two are equivalent when $d = 2$.)*

We consider the case of points on $\sigma_3(v_d(\mathbb{P}W))\backslash\sigma_2(v_d(\mathbb{P}W))$. We cannot have three distinct limiting points x_1, x_2, x_3 with $\dim\langle x_1, x_2, x_3\rangle < 3$ unless at least two of them coincide because there are no trisecant lines to $v_d(\mathbb{P}W)$. (For any variety $X \subset \mathbb{P}V$ with ideal generated in degree two, any trisecant line of X is contained in X, and Veronese varieties $v_d(\mathbb{P}W) \subset \mathbb{P}S^dW$ are cut out by quadrics but contain no lines.)

Write our curves as

$$\begin{aligned} x(t) &= (x_0 + tx_1 + t^2x_2 + t^3x_3 + \cdots)^d \\ &= x_0^d + t(dx_0^{d-1}x_1) + t^2\left(\binom{d}{2}x_0^{d-2}x_1^2 + dx_0^{d-1}x_2\right) \\ &\quad + t^3\left(\binom{d}{3}x_0^{d-3}x_1^3 + d(d-1)x_0^{d-2}x_1x_2 + dx_0^{d-1}x_3\right) + \cdots \end{aligned}$$

and similarly for $y(t)$, $z(t)$.

Case 1: two distinct limit points x_0^d, z_0^d, with $y_0 = x_0$. (We can always rescale to have equality of points rather than just collinearity since we are working in projective space.) When we expand the Taylor series, assuming $d > 2$ (since the $d = 2$ case is well understood and different), the coefficient of t (ignoring constants which disappear when projectivizing) is

$$x_0^{d-1}(x_1 - y_1) \wedge x_0^d \wedge z_0^d$$

which can be zero only if the first term is zero, i.e., $x_1 \equiv y_1 \mod x_0$. If this happens, examining (8) we see the second order term is of the form

$$x_0^{d-1}(x_2 - y_2 + \lambda x_1) \wedge x_0^d \wedge z_0^d.$$

Similarly if this term vanishes, the t^3 term will still be of the same nature. Inductively, if the lowest nonzero term is t^k then for each $j < k$, $y_j = x_j \mod (x_0, \dots, x_{j-1})$, and the coefficient of the t^k term is (up to a constant factor)

$$x_0^{d-1}(x_k - y_k + \ell) \wedge x_0^d \wedge z_0^d$$

where ℓ is a linear combination of $x_0, \dots, x_{k-1}$. We rewrite this as $x^{d-1}y \wedge x^d \wedge z^d$. If $\dim\langle z, x, y\rangle < 3$ we are reduced to a point of $\sigma_3(v_d(\mathbb{P}^1))$ and can appeal to Theorem 4.1. If the span is three dimensional then any point in the plane $[x^{d-1}y \wedge x^d \wedge z^d]$ can be put in the normal form $x^{d-1}w + z^d$.

Case 2: One limit point $x_0 = y_0 = z_0 = z$. The t coefficient vanishes and the t^2 coefficient is (up to a constant factor)

$$x_0^{d-1}(x_1 - y_1) \wedge x_0^{d-1}(y_1 - z_1) \wedge x_0^d$$

which can be rewritten as $x^{d-1}y \wedge x^{d-1}z \wedge x^d$. If this expression is nonzero then any point in the plane $[x^{d-1}y \wedge x^{d-1}z \wedge x^d]$ lies in $\sigma_2(v_d(\mathbb{P}^1))$. So we thus assume the t^2 coefficent vanishes. Then $y_1 - z_1$, $x_1 - y_1$, and x_0 are linearly dependent; a straightforward calculation shows that the t^3 coefficient is

$$x_0^d \wedge x_0^{d-1}(y_1 - x_1) \wedge (x_0^{d-1}\ell + (\lambda^2 + \lambda)(y_1 - x_1)^2),$$

where ℓ is a linear combination of $x_0, \dots, z_2$. We rewrite this as $x^d \wedge x^{d-1}y \wedge (x^{d-1}\ell + \mu x^{d-2}y^2)$. If $\mu = 0$, every point in the plane $[x^d \wedge x^{d-1}y \wedge x^{d-1}\ell]$ lies in $\sigma_2(v_d(\mathbb{P}^1))$, so we apply Theorem 4.1. If $\mu \neq 0$ and $x^d \wedge x^{d-1}y \wedge (x^{d-1}\ell + \mu x^{d-2}y^2) = 0$, then x, y are linearly dependent; then one considers higher powers of t. One can argue that the lowest nonzero term always has the form $x^d \wedge x^{d-1}y \wedge (x^{d-1}\ell + \mu x^{d-2}y^2)$.

Thus our point lies in a plane of the form $[x^d \wedge x^{d-1}y \wedge (x^{d-1}\ell + \mu x^{d-2}y^2)]$.

Theorem 10.2. *There are three types of points $\phi \in S^3W$ of border rank three with $\dim\langle\phi\rangle = 3$. They have the following normal forms:*

limiting curves	normal form	R
x^d, y^d, z^d	$x^d + y^d + z^d$	3
$x^d, (x+ty)^d, z^d$	$x^{d-1}y + z^d$	$d \le R \le d+1$
$x^d, (x+ty)^d, (x+2ty+t^2z)^d$	$x^{d-2}y^2 + x^{d-1}z$	$d \le R \le 2d-1$

The upper bounds on ranks come from computing the sum of the ranks of the terms. The lower bounds on ranks are attained by specialization to $S^d\mathbb{C}^2$. We remark that when $d = 3$, the upper bounds on rank are attained in both cases.

Corollary 10.3. *Let $\phi \in S^dW$ with $\underline{R}(\phi) = 3$. If $R(\phi) > 3$, then $2d-1 \ge R(\phi) \ge d-1$ and only three values occur, one of which is $d-1$.*

Proof. The only additional cases occur if $\dim\langle\phi\rangle = 2$ which are handled by Theorem 4.1. □

Remark 10.4. Even for higher secant varieties, $x_1^d \wedge \cdots \wedge x_r^d$ cannot be zero if the x_j are distinct points, even if they lie on a $\mathbb{P}^1$, as long as $d \ge r$. This is because a hyperplane in S^dW corresponds to a (defined up to scale) homogeneous polynomial of degree d on W. Now take $W = \mathbb{C}^2$. No homogeneous polynomial of degree d vanishes at $d+1$ distinct points of $\mathbb{P}^1$, thus the image of any $d+1$ distinct points under the d-th Veronese embedding cannot lie on a hyperplane.

Theorem 10.5. *There are six types of points of border rank four in S^dW, $d > 2$, whose span is 4 dimensional. They have the following normal forms:*

limiting curves	normal form	R
x^d, y^d, z^d, w^d	$x^d + y^d + z^d + w^d$	4
$x^d, (x+ty)^d, z^d, w^d$	$x^{d-1}y + z^d + w^d$	$d \le R \le d+2$
$x^d, (x+ty)^d, z^d, (z+tw)^d$	$x^{d-1}y + z^{d-1}w$	$d \le R \le 2d$
$x^d, (x+ty)^d, (x+ty+t^2z)^d, (x+t^2z)^d$	$x^{d-2}yz$	$d \le R \le 2d-2$
$x^d, (x+ty)^d, (x+ty+t^2z)^d, w^d$	$x^{d-2}y^2 + x^{d-1}z + w^d$	$d \le R \le 2d$
$x^d, (x+ty)^d, (x+ty+t^2z)^d, (x+ty+t^2z+t^3w)^d$	$x^{d-3}y^3 + x^{d-2}z^2 + x^{d-1}w$	$d \le R \le 3d-3$

For $\sigma_5(v_d(\mathbb{P}W))$, we get a new phenomenon when $d = 3$ because $\dim S^3\mathbb{C}^2 = 4 < 5$. We can have 5 curves a, b, c, d, e, with $a_0, \dots, e_0$ all lying in a $\mathbb{C}^2$, but otherwise general, so $\dim\langle a_0^3, \dots, e_0^3\rangle = 4$. Thus the t term will be of the form $a_0^3 \wedge b_0^3 \wedge c_0^3 \wedge d_0^3 \wedge (s_1a_0^2a_1 + \cdots + s_4d_0^2d_1 - e_0^2e_1)$. Up to scaling we can give $\mathbb{C}^2$ linear coordinates x, y so that $a_0 = x$, $b_0 = y$, $c_0 = x+y$, $d_0 = x + \lambda y$ for some λ. Then, independent of e_0, the limiting plane will be contained in

$$\langle x^3, y^3, (x+y)^3, (x+\lambda y)^3, x^2\alpha, xy\beta, y^2\gamma\rangle$$

for some $\alpha, \beta, \gamma \in W$ (depending on $a_1, \dots, e_1$). Any point contained in the plane is of the form $x^2u + y^2v + xyz$ for some $u, v, z \in W$.

Theorem 10.6. *There are seven types of points of border rank five in S^dW whose span is five dimensional when $d > 3$, and eight types when $d = 3$. Six of the types are obtained by adding a term of the form u^d to a point of border rank four, the seventh has the normal form $x^{d-4}u + x^{d-3}y^3 + x^{d-2}z^2 + x^{d-1}w$, and the eighth type, which occurs when $d = 3$, has normal form $x^2u + y^2v + xyz$.*

Remark 10.7. By dimension count, we expect to have normal forms of elements of $\sigma_r(v_d(\mathbb{P}^{n-1}))$ as long as $r \le n$ because $\dim \sigma_r(v_d(\mathbb{P}^{n-1})) \le rn-1$ and $\dim GL_n = n^2$.

11. Monomials

11.1. **Limits of highest possible osculation.** Let $x(t) \subset W$ be a curve, write $x_0 = x(0)$, $x_1 = x'(0)$ and $x_j = x^{(j)}(0)$. Consider the corresponding curve $y(t) = x(t)^d$ in $\hat{v}_d(\mathbb{P}W)$ and note that

$$\begin{aligned}
y(0) &= x_0^d \\
y'(0) &= dx_0^{d-1}x_1 \\
y''(0) &= d(d-1)x_0^{d-2}x_1^2 + dx_0^{d-1}x_2 \\
y^{(3)}(0) &= d(d-1)(d-2)x_0^{d-3}x_1^3 + 3d(d-1)x_0^{d-2}x_1x_2 + dx_0^{d-1}x_3 \\
y^{(4)}(0) &= d(d-1)(d-2)(d-3)x_0^{d-4}x_1^4 + 6d(d-1)(d-2)x_0^{d-3}x_1^2x_2 + 3d(d-1)x_0^{d-2}x_2^2 \\
&\quad + 4d(d-1)x_0^{d-2}x_1x_3 + dx_0^{d-1}x_4 \\
y^{(5)}(0) &= d(d-1)(d-2)(d-3)(d-4)x_0^{d-5}x_1^5 + 9d(d-1)(d-2)(d-3)x_0^{d-4}x_1^3x_2 \\
&\quad + 10d(d-1)(d-2)x_0^{d-3}x_1^2x_3 + 15d(d-1)(d-2)x_0^{d-3}x_1x_2^2 \\
&\quad + 4d(d-1)x_0^{d-2}x_2x_3 + 5d(d-1)x_0^{d-2}x_1x_4 + dx_0^{d-1}x_5 \\
&\ \ \vdots
\end{aligned}$$

At r derivatives, we get a sum of terms

$$x_0^{d-s}x_1^{a_1}\cdots x_p^{a_p}, \quad a_1 + 2a_2 + \cdots + pa_p = r, \quad s = a_1 + \cdots + a_p.$$

In particular, $x_0x_1\cdots x_{d-1}$ appears for the first time at the $1+2+\cdots+(d-1) = \binom{d}{2}$ derivative.

11.2. **Bounds for monomials.** Write $\mathbf{b} = (b_1, \dots, b_m)$. Let $S_{\mathbf{b},\delta}$ denote the number of distinct m-tuples $(a_1, \dots, a_m)$ satisfying $a_1 + \cdots + a_m = \delta$ and $0 \le a_j \le b_j$. Adopt the notation that $\binom{a}{b} = 0$ if $b > a$ and is the usual binomial coefficient otherwise. We thank L. Matusevich for the following expression:

Proposition 11.1. *Write $I = (i_1, i_2, \dots, i_k)$ with $i_1 \le i_2 \le \cdots \le i_k$. Then*

$$S_{\mathbf{b},\delta} = \sum_{k=0}^{m}(-1)^k \left[\sum_{|I|=k} \binom{\delta + m - k - (b_{i_1} + \cdots + b_{i_k})}{m}\right].$$

Proof. The proof is safely left to the reader. (It is a straightforward inclusion-exclusion counting argument, in which the kth term of the sum counts the m-tuples with $a_j \ge b_j + 1$ for at least k values of the index j.) For those familiar with algebraic geometry, note that $S_{\mathbf{b},\delta}$ is the Hilbert function of the variety defined by the monomials $x_1^{b_1+1}, \dots, x_m^{b_m+1}$. $\square$

For $\mathbf{b} = (b_1, \dots, b_m)$, consider the quantity

$$T_{\mathbf{b}} := \prod_{i=1}^{m}(1 + b_i).$$

$T_{\mathbf{b}}$ counts the number of tuples $(a_1, \dots, a_m)$ satisfying $0 \le a_j \le b_j$ (with no restriction on $a_1 + \cdots + a_m$).

Theorem 11.2. *Let $b_0 \geq b_1 \geq \cdots \geq b_n$ and write $d = b_0 + \cdots + b_n$. Then*

$$S_{(b_0,b_1,\dots,b_n),\lfloor \frac{d}{2} \rfloor} \leq \underline{R}(x_0^{b_0} x_1^{b_1} \cdots x_n^{b_n}) \leq T_{(b_1,\dots,b_n)}.$$

Proof. Let $\phi = x_0^{b_0} \cdots x_n^{b_n}$. The lower bound follows from considering the image of $\phi_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}$, which is

$$\phi_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}(S^{\lceil \frac{d}{2} \rceil}\mathbb{C}^{n+1}) = \left\langle x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} \,\middle|\, 0 \leq a_j \leq b_j,\, a_0 + a_1 + \cdots + a_n = \left\lfloor \frac{d}{2} \right\rfloor \right\rangle$$

whose dimension is $S_{(b_0,b_1,\dots,b_n),\lfloor \frac{d}{2} \rfloor}$.

We show the upper bound as follows. Let

$$F_{\mathbf{b}}(t) = \bigwedge_{s_1=0}^{b_1} \cdots \bigwedge_{s_n=0}^{b_n} (x_0 + t^1 \lambda_{1,s_1} x_1 + t^2 \lambda_{2,s_2} x_2 + \cdots + t^n \lambda_{n,s_n} x_n)^d \tag{9}$$

where the $\lambda_{i,s}$ are chosen sufficiently generally. We may take each $\lambda_{i,0} = 0$ and each $\lambda_{i,1} = 1$ if we wish. For $t \neq 0$, $[F_{\mathbf{b}}(t)]$ is a plane spanned by $T_{\mathbf{b}}$ points in $v_d(\mathbb{P}W)$. We claim $x_0^{b_0} \cdots x_n^{b_n}$ lies in the plane $\lim_{t\to 0}[F_{\mathbf{b}}(t)]$, which shows $\underline{R}(x_0^{b_0} \cdots x_n^{b_n}) \leq T_{\mathbf{b}}$. In fact, we claim

$$\lim_{t\to 0}[F_{\mathbf{b}}(t)] = \left[\bigwedge_{a_1=0}^{b_1} \cdots \bigwedge_{a_n=0}^{b_n} x_0^{d-(a_1+\cdots+a_n)} x_1^{a_1} \cdots x_n^{a_n} \right] \tag{10}$$

so $x_0^{b_0} \cdots x_n^{b_n}$ occurs precisely as the last member of the spanning set for the limit plane.

For an n-tuple $I = (a_1, \dots, a_n)$ and an n-tuple $(p_1, \dots, p_n)$ satisfying $0 \leq p_i \leq b_i$, let

$$c_{(p_1,\dots,p_n)}^{(a_1,\dots,a_n)} = \lambda_{1,p_1}^{a_1} \cdots \lambda_{n,p_n}^{a_n},$$

the coefficient of $x_1^{a_1} \cdots x_n^{a_n} x_0^{d-(a_1+\cdots+a_n)}$ in $(x_0 + t\lambda_{1,p_1} x_1 + \cdots + t^n \lambda_{n,p_n} x_n)^d$, omitting binomial coefficients. Choose an enumeration of the n-tuples $(p_1, \dots, p_n)$ satisfying $0 \leq p_i \leq b_i$; say, in lexicographic order. Then given n-tuples $I_1, \dots, I_T$, the coefficient of the term

$$x^{I_1} \wedge \cdots \wedge x^{I_T}$$

in $F_{\mathbf{b}}(t)$ is the product $\prod_{j=1}^T c_j^{I_j}$, omitting binomial coefficients. We may interchange the x^{I_j} so that $I_1 \leq \cdots \leq I_T$ in some order, say lexicographic. Then the total coefficient of $x^{I_1} \wedge \cdots \wedge x^{I_T}$ is the alternating sum of the permuted products,

$$\sum_{\pi} (-1)^{|\pi|} c_{\pi(j)}^{I_j},$$

(summing over all permutations π of $\{1, \dots, T\}$) times a product of binomial coefficients (which we henceforth ignore). This sum is the determinant of the $T \times T$ matrix $C := (c_i^{I_j})_{i,j}$.

We may assume the monomials $x^{I_1}, \dots, x^{I_T}$ are all distinct (otherwise the term $x^{I_1} \wedge \cdots \wedge x^{I_T}$ vanishes identically). Suppose some monomial in $x_2, \dots, x_n$ appears with more than $b_1 + 1$ different powers of x_1; without loss of generality,

$$x^{I_1} = x_0^{p-a_1} x_1^{a_1} r, \quad \dots \quad , \quad x^{I_{b_1+2}} = x_0^{p-a_{b_1+2}} x_1^{a_{b_1+2}} r,$$

where r is a monomial in $x_2, \dots, x_n$, and $p = d - \deg(r)$. The matrix $(\lambda_{1,i}^{a_j})_{i,j}$ has size $(b_1 + 1) \times (b_1 + 2)$. The dependence relation among the columns of this matrix holds for the first $b_1 + 2$ columns of C, since (in these columns) each row is multiplied by the same factor r.

More generally, if r is any monomial in $(n - 1)$ of the variables, say $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, then $x_i^a r$ can occur for at most $b_i + 1$ distinct values of the exponent a. The lowest power of t occurs when the values of a are $a = 0, 1, \dots$. In particular $x_i^a r$ only occurs for $a \leq b_i$.

Therefore, if a term $x^{I_1} \wedge \cdots \wedge x^{I_T}$ has a nonzero coefficient in $F_{\mathbf{b}}(t)$ and occurs with the lowest possible power of t, then in every single x^{I_j}, each x_i occurs to a power $\leq b_i$. The only way the x^{I_j} can be distinct is for it to be the term in the right hand side of (10). This shows that no other term with the same or lower power of t survives in $F_{\mathbf{b}}(t)$; we only have to show that the term we want has a nonzero coefficient. For this term C is a tensor product,

$$C = (\lambda^j_{1,i})^{(b_1,b_1)}_{(i,j)=(0,0)} \otimes \cdots \otimes (\lambda^j_{n,i})^{(b_n,b_n)}_{(i,j)=(0,0)},$$

and each matrix on the right hand side is nonsingular since they are Vandermonde matrices and the $\lambda_{p,i}$ are distinct. Therefore the coefficient of the term in (10) is $\det C \neq 0$. □

For example

$$\begin{aligned} F_{(b)}(t) &= x_0^d \wedge \bigwedge_{s=1}^{b} (x_0 + t\lambda_s x_1)^d \\ &= t^{\binom{b+1}{2}} \left[x_0^d \wedge \left(\sum (-1)^s \lambda_s\right) x_0^{d-1}x_1 \wedge \sum (-1)^{s+1}\lambda_s^2 x_0^{d-2}x_1^2 \wedge \cdots \wedge \sum (-1)^{s+b}\lambda_s^b x_0^{d-b}x_1^b \right] \\ &\quad + O\big(t^{\binom{b+1}{2}+1}\big) \end{aligned}$$

and (with each $\lambda_{i,s} = 1$)

$$\begin{aligned} F_{(1,1)}(t) &= x_0^d \wedge (x_0 + tx_1)^d \wedge (x_0 + t^2x_2)^d \wedge (x_0 + tx_1 + t^2x_2)^d \\ &= x_0^d \wedge \left(x_0^d + dtx_0^{d-1}x_1 + \binom{d}{2} t^2 x_0^{d-2}x_1^2 + \cdots \right) \\ &\quad \wedge \left(x_0^d + dt^2 x_0^{d-1}x_2 + \cdots \right) \\ &\quad \wedge \left(x_0^d + dtx_0^{d-1}x_1 + t^2\left(\binom{d}{2} x_0^{d-2}x_1^2 + dx_0^{d-1}x_2\right) \right. \\ &\qquad \left. + t^3\left(\binom{d}{3} x_0^{d-3}x_1^3 + d(d-1)x_0^{d-2}x_1x_2\right) + \cdots \right) \\ &= t^6 \left(x_0^d \wedge dx_0^{d-1}x_1 \wedge dx_0^{d-1}x_2 \wedge d(d-1)x_0^{d-2}x_1x_2 \right) + O(t^7). \end{aligned}$$

Theorem 11.3. *Let $b_0 \geq b_1 + \cdots + b_n$. Then $\underline{R}(x_0^{b_0}x_1^{b_1}\cdots x_n^{b_n}) = T_{(b_1,\ldots,b_n)}$.*

Theorem 11.3 is an immediate consequence of Theorem 11.2 and the following lemma:

Lemma 11.4. *Let $\mathbf{a} = (a_1, \ldots, a_n)$. Write $\mathbf{b} = (a_0, \mathbf{a})$ with $a_0 \geq a_1 + \cdots + a_n$. Then for $a_1 + \cdots + a_n \leq \delta \leq a_0$, $S_{\mathbf{b},\delta}$ is independent of δ and in fact $S_{\mathbf{b},\delta} = T_{\mathbf{a}}$.*

Proof. The right hand side $T_{\mathbf{a}}$ counts n-tuples $(e_1, \ldots, e_n)$ such that $0 \leq e_j \leq a_j$. To each such tuple we associate the $(n+1)$-tuple $(\delta - (e_1 + \cdots + e_n), e_1, \ldots, e_n)$. Since

$$0 \leq \delta - (a_1 + \cdots + a_n) \leq \delta - (e_1 + \cdots + e_n) \leq \delta \leq a_0,$$

this is one of the tuples counted by the left hand side $S_{\mathbf{b},\delta}$, establishing a bijection between the sets counted by $S_{\mathbf{b},\delta}$ and $T_{\mathbf{a}}$. □

In particular,

Corollary 11.5. *Write $d = a + n$, and consider the monomial $\phi = x_0^a x_1 \cdots x_n$. If $a \geq n$, then $\underline{R}(x_0^a x_1 \cdots x_n) = 2^n$. Otherwise,*

$$\binom{n}{\lfloor \frac{d}{2} \rfloor - a} + \binom{n}{\lfloor \frac{d}{2} \rfloor - a + 1} + \cdots + \binom{n}{\lfloor \frac{d}{2} \rfloor} \leq \underline{R}(x_0^a x_1 \cdots x_n) \leq 2^n.$$

Proof. The right hand inequality follows as $T_{(1,\dots,1)} = 2^n$. To see the left hand inequality, for $0 \le k \le a$, let $e = \lfloor \frac{d}{2} \rfloor - a + k$. Then $\binom{n}{e}$ is the number of monomials of the form $x_0^{\lfloor d/2 \rfloor - e} x_{i_1} \cdots x_{i_e}$, $1 \le i_1 < \cdots < i_e \le n$ and $S_{(a,1,\dots,1),\lfloor \frac{d}{2} \rfloor}$ is precisely the total number of all such monomials for all values of e. □

Proposition 11.6.
$$\binom{n}{\lfloor n/2 \rfloor} + \lceil n/2 \rceil - 1 \le R(x_1 \cdots x_n) \le 2^{n-1},$$
$$\binom{n}{\lfloor n/2 \rfloor} \le \underline{R}(x_1 \cdots x_n) \le 2^{n-1}.$$

Proof. Write $\phi = x_1 \cdots x_n$. First,
$$\phi = \frac{1}{2^{n-1} n!} \sum_{\epsilon \in \{-1,1\}^{n-1}} (x_1 + \epsilon_1 x_2 + \cdots + \epsilon_{n-1} x_n)^n \epsilon_1 \cdots \epsilon_{n-1},$$
a sum with 2^{n-1} terms, so $R(\phi) \le 2^{n-1}$.

Now, for $1 \le a \le n-1$, the image of $\phi_{a,n-a}$ is spanned by the monomials $x_{i_1} \cdots x_{i_a}$, $1 \le i_1 < \cdots < i_a \le n$. So $\operatorname{rank} \phi_{a,n-a} = \binom{n}{a}$. Thus $\underline{R}(\phi) \ge \binom{n}{\lfloor n/2 \rfloor}$. The set Σ_a consists of those points $p \in \mathbb{P}W^* \cong \mathbb{P}^{n-1}$ at which (at least) $a+1$ of the coordinate functions vanish. So $\dim \Sigma_a = n - a - 2$. Therefore $R(\phi) > \binom{n}{a} + n - a - 2$, for $1 \le a \le n-1$. This quantity is maximized at $a = \lfloor n/2 \rfloor$. □

To give a sense of how these bounds behave, we illustrate with the following table for bounds on the ranks and border ranks of $x_1 \cdots x_n$.

n	1	2	3	4	5	6	7	8	9	10
upper bound for $R(x_1 \cdots x_n)$	1	2	4	8	16	32	64	128	256	512
lower bound for $R(x_1 \cdots x_n)$	1	2	4	7	12	22	38	73	130	256
lower bound for $\underline{R}(x_1 \cdots x_n)$	1	2	3	6	10	20	35	70	126	252

For $n < 4$ the upper and lower bounds agree. Here is the next case:

Proposition 11.7. $R(x_1x_2x_3x_4) = 8$.

Proof. Suppose $R(x_1x_2x_3x_4) = 7$. Write $\phi = x_1x_2x_3x_4 = \eta_1^4 + \cdots + \eta_7^4$ with the $[\eta_i] \in \mathbb{P}W$ distinct points. Let $L = \{p \in S^2W^* \mid p(\eta_i) = 0,\ i = 1, \dots, 7\}$, so $\mathbb{P}L \subset \mathbb{P}\,\mathrm{Rker}\,\phi_{2,2}$. We have $\dim \mathbb{P}L \ge \dim \mathbb{P}S^2W^* - 7 = 2$. On the other hand, $\mathbb{P}L$ is contained in $\mathbb{P}\,\mathrm{Rker}\,\phi_{2,2}$ and disjoint from $\mathbb{P}\,\mathrm{Rker}\,\phi_{2,2} \cap v_2(\mathbb{P}W^*) \cong \Sigma_2$, so $\dim \mathbb{P}L \le 2$ (as in the proof of Theorem 1.3).

We will show that there are six reducible quadrics in $\mathbb{P}L$, and they restrict the η_i in such a way to imply a contradiction.

Observe that
$$R(\phi - x_1^4) > \operatorname{rank}(\phi - x_1^4)_{2,2} + \dim \Sigma_2(\phi - x_1^4) = 7 + 0.$$
If one of the η_i were (a scalar multiple of) x_1 then we would have $R(\phi - x_1^4) \le R(\phi) - 1 < 7$. By the same argument for $x_2, \dots, x_4$, all 11 of the points $[x_i], [\eta_j]$ are distinct.

Let $\alpha_1, \dots, \alpha_4$ be the dual basis of W^* to $x_1, \dots, x_4$. $\mathbb{P}\,\mathrm{Rker}\,\phi_{2,2}$ is spanned by $\{[\alpha_1^2], \dots, [\alpha_4^2]\} = \mathbb{P}\,\mathrm{Rker}\,\phi_{2,2} \cap v_2(\mathbb{P}W^*)$. The reducible quadrics in $\mathbb{P}\,\mathrm{Rker}\,\phi_{2,2}$ are precisely the elements $[p\alpha_i^2 + q\alpha_j^2]$, $i \ne j$, that is, the lines which form the edges of the tetrahedron with vertices at the $[\alpha_i^2]$. By a dimension count, L intersects these lines. Since L is a linear subspace, it intersects the tetrahedron at precisely six points, which are not the vertices. This shows there are precisely six reducible quadrics passing through the $[\eta_i]$.

Denote them $Q_{12}, \dots, Q_{34}$, where Q_{ij} spans $L \cap \langle \alpha_i^2, \alpha_j^2 \rangle$. Up to scaling the Q_{ij}, there are constants $b_1, \dots, b_4$ such that $Q_{ij} = b_i\alpha_i^2 - b_j\alpha_j^2$. (Indeed, writing each $Q_{1j} = \alpha_1^2 - b_j\alpha_j^2$, Q_{jk}

must be a scalar times $Q_{1k} - Q_{1j}$, from which the claim follows.) The b_i are nonzero, so we may rescale coordinates so each $b_i = 1$.

Then up to scalar multiple each $\eta_i = x_1 \pm x_2 \pm x_3 \pm x_4$. Solving for the coefficients c_i in $x_1x_2x_3x_4 = c_1\eta_1^4 + \cdots + c_7\eta_7^4$ shows there are no solutions. Equivalently, let $\eta_1, \dots, \eta_8$ be all 8 of the points $x_1 \pm x_2 \pm x_3 \pm x_4$. There is no solution for c_i in $x_1x_2x_3x_4 = c_1\eta_1^4 + \cdots + c_8\eta_8^4$ with one of the $c_i = 0$. □

Remark 11.8. The singular quadrics in $\mathbb{P}\,\mathrm{Rker}\,\phi_{2,2}$ are those of the form $[p\alpha_{i_1}^2 + q\alpha_{i_2}^2 + r\alpha_{i_3}^2]$, where $\{i_1, i_2, i_3\} \subset \{1, 2, 3, 4\}$ which correspond to the faces of the tetrahedron spanned by $[\alpha_1^2], \dots, [\alpha_4^2]$. Each such quadric is singular at $[x_{i_4}]$, where $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. It would be interesting to see if considering these singular quadrics, instead of the reducible quadrics, yields a simpler proof that $R(x_1x_2x_3x_4) > 7$.

Proposition 11.9. *$R(x^2yz) = 6$ and $\underline{R}(x^2yz) = 4$.*

Proof. Let $\phi = x^2yz$. Then $\langle\phi\rangle = W$, so $R(\phi) \geq \mathrm{rank}\,\phi_{1,2} = \dim W = 3$. The singular set $\Sigma_1 = \{x = 0\} \cup \{y = z = 0\}$ has dimension 1. Therefore $R(\phi) \geq 3 + 1 + 1 = 5$. In the other direction,

$$\phi = x^2\Big(\frac{y+z}{2}\Big)^2 - x^2\Big(\frac{y-z}{2}\Big)^2,$$

and note that $R(a^2b^2) = 3$ by Theorem 4.1 or explicitly

$$a^2b^2 = (a+b)^4 + \omega(a+\omega b)^4 + \omega^2(a+\omega^2 b)^4, \qquad \omega = e^{2\pi i/3}$$

so that $R(\phi) \leq 3 + 3 = 6$. Thus $5 \leq R(\phi) \leq 6$ and $\underline{R}(\phi) = 4$ by Theorem 11.3.

We will show that in fact $R(\phi) = 6$, following a suggestion provided to us by Bruce Reznick. Suppose that $R(\phi) = 5$, with $\phi = \eta_1^4 + \cdots + \eta_5^4$, for some distinct $[\eta_i] \in \mathbb{P}W = \mathbb{P}^2$. Let $L := \mathbb{P}\{p \in S^2W^* \mid p(\eta_i) = 0, 1 \leq i \leq 5\}$. The proof of Theorem 1.3 shows $\dim L = 0$, i.e., L consists of exactly one point, so the $[\eta_i]$ lie on a unique conic Q in the projective plane. In particular, no four of the $[\eta_i]$ are collinear. One checks that $R(x^2yz - \lambda x^4) \geq 5$ by Theorem 1.3, for all λ, and so no $[\eta_i] = [x]$.

The conic Q is an element of $\mathbb{P}\,\mathrm{Rker}\,\phi_{2,2}$, which one finds is spanned by β^2 and γ^2. Therefore Q factors, $Q = (c\beta - d\gamma)(c\beta + d\gamma)$. We have $c, d \neq 0$ (or else all five $[\eta_i]$ are collinear).

Therefore exactly three of the $[\eta_i]$ lie on one of the lines of Q and exactly two lie on the other line. Up to reordering, we have $\eta_i = s_ix + t_i(dy + cz)$ for $i = 1, 2, 3$ and $\eta_i = s_ix + t_i(dy - cz)$ for $i = 4, 5$. The subsitution $z \to \frac{-d}{c}y$ takes the equation

$$\phi = x^2yz = \eta_1^4 + \cdots + \eta_5^4$$

to

$$\frac{-d}{c}x^2y^2 = (s_1^4 + s_2^4 + s_3^4)x^4 + \overline{\eta}_4^4 + \overline{\eta}_5^4$$

where $\overline{\eta}_4, \overline{\eta}_5$ are linear forms in x, y. Multiplying by scalar factors, this gives an expression of $x^2y^2 - Ax^4$ as a sum of two fourth powers. But we have $R(x^2y^2 - Ax^4) \geq 3$ for all A; indeed, the symmetric flattening $(x^2y^2 - Ax^4)_{2,2}$ has rank 3 already.

This contradiction shows $R(\phi) > 5$. □

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$$
\begin{aligned}
xy(x+y) &= \frac{1}{3\sqrt{3}i}\Big((\omega x-y)^3-(\omega^2x-y)^3\Big) \qquad (\omega=e^{2\pi i/3})\\
x^2y &= \frac{1}{6}\Big((x+y)^3-(x-y)^3-2y^3\Big)\\
y^2z-x^3 &= \frac{1}{6}\Big((y+z)^3+(y-z)^3-2z^3\Big)-x^3\\
xyz &= \frac{1}{24}((x+y+z)^3-(-x+y+z)^3-(x-y+z)^3-(x+y-z)^3)\\
x(x^2+yz) &= \frac{1}{288}\Big((6x+2y+z)^3+(6x-2y-z)^3\\
&\qquad -\sqrt{3}(2\sqrt{3}x-2y+z)^3-\sqrt{3}(2\sqrt{3}x+2y-z)^3\Big)\\
y^2z-x^3-xz^2 &= \frac{-1}{12\sqrt{3}}\Big((3^{1/2}x+3^{1/4}iy+z)^3+(3^{1/2}x-3^{1/4}iy+z)^3\\
&\qquad +(3^{1/2}x+3^{1/4}y-z)^3+(3^{1/2}x-3^{1/4}y-z)^3\Big)\\
y^2z-x^3-z^3 &= \frac{1}{6\sqrt{3}i}\Big((2\omega z-(y-z))^3-(2\omega^2z-(y-z))^3\Big)-x^3\\
y^2z-x^3-axz^2-bz^3 &= z(y-b^{1/2}z)(y+b^{1/2}z)-x(x-a^{1/2}iz)(x+a^{1/2}iz)\\
&= \frac{1}{6b^{1/2}\sqrt{3}i}\Big((2\omega b^{1/2}z-(y-b^{1/2}z))^3-(2\omega^2b^{1/2}z-(y-b^{1/2}z))^3\Big)\\
&\quad -\frac{1}{6\sqrt{3}i}\Big((\omega(x-a^{1/2}iz)-(x+a^{1/2}iz))^3-(\omega^2(x-a^{1/2}iz)-(x+a^{1/2}iz))^3\Big)\\
y(x^2+yz) &= (x-y)(x+y)y+y^2(y+z)\\
&= \frac{1}{6\sqrt{3}i}\Big((2\omega y-(x-y))^3-(2\omega^2y-(x-y))^3\Big)+\frac{1}{6}\Big((2y+z)^3+z^3-2(y+z)^3\Big)
\end{aligned}
$$

TABLE 3. Upper bounds on ranks of plane cubic forms.