A SUMMARY OF MY RESEARCH (MARCH 2016)

J.M. LANDSBERG

I have wide research interests: Cartan style differential geometry, classically influenced algebraic geometry, the geometry of homogeneous varieties, categorical generalizations of Lie algebras, the geometry and application of tensors, and most recently in algebraic complexity theory. I am the co-author (with Ivey) of *Cartan for beginners* [14] (second edition to appear in 2016), the author of *Tensors: geometry and applications* [29] and the forthcoming *Geometry and Complexity Theory* to be published by Cambridge, 2016 (draft available at http://www.math.tamu.edu/~jml/simonsclass.pdf). What follows is a summary of results organized by themes.

Complexity theory. Algebraic complexity theory deals with determining the computational cost of common operations such as multiplying matrices or computing the discrete Fourier transform. There are three aspects: *lower bounds* – showing there does not exist a "faster" algorithm, *upper bounds* – showing there does, and *algorithms*– actually finding fast algorithms. For example, in 1969 Strassen discovered the usual algorithm for multiplying matrices that everyone uses is not optimal, and since then computer scientists have conjectured that as n goes to infinity, it becomes almost as easy to multiply two $n \times n$ matrices as it is to add them. For the central problem of determining the complexity of matrix multiplication, there had been no progress on lower bounds for 25 years until the breakthrough papers [26, 54]. A second central problem is permanent v. determinant, Valiant's algebraic analog of **P** v. **NP**. Here I have several important results, including the first exponential separation of permanent from determinant in any restricted model [52]. Recently I have been exploring the use of algebraic geometry to prove upper bounds and find practical algorithms, e.g., [47].

- First exponential separation of permanent from determinant in any restricted model [52] (with Ressayre).
- First improvement on the lower bound of the border rank the matrix multiplication tensor in over 25 years [54] (with Ottaviani), as well as current world record [44] (with Michalek).
- Solved longstanding (25 years) problem of determining the border rank of the 2×2 matrix multiplication tensor [26]. Gave second proof using completely different methods (with Hauenstein and Ikenmeyer) [10].
- First improvement in the lower bound of the rank the matrix multiplication tensor in over 25 years [30]. Current world record was obtained by students working under my direction [60].
- Establishment of mathematical foundations of Geometric Complexity Theory, an approach to Valiant's conjecture via algebraic geometry and representation theory proposed in [61, 62] by Mulmuley and Sohoni, [5] (with Burgisser, Manivel and Weyman) and [31].
- Current world record for the GCT separation of permanent from determinant [53] (with Manivel and Ressayre).

- Proved the celebrated method of shifted partial derivatives, despite hopes of computer scientists, cannot be used to prove any significant separation of permanent from determinant [7] (with Efremenko, Schenck, and Weyman).
- Geometric study of algorithms of Smirnov and Bini et. al. [47] (with Ryder).
- Extremal characterizations of the Coppersmith-Winograd tensor used in [6, 58] to prove the best upper bounds for the exponent of matrix multiplication that indicate it will be difficult to find a better tensor for the Coppersmith-Winograd method as proposed in [1], [43, 44]. (with Michalek).
- Establishment of methods from algebraic geometry to prove matrix rigidity, including the first non-classical equations for non-rigidity [8] (with Gesmundo, Hauenstein, and Ikenmeyer).
- Established conditions for modules of obstructions in the sense of GCT to be useful, which was used by Ikenmeyer-Panova [11] to disprove a central conjecture in GCT [15] (with Kadish).
- Determined geometric formulation and generalizations of Valiant's holographic algorithms [45] (with Morton and Norine) and [28].

Tensors, Waring problems, and applied tensor questions. Tensors have long been used in theoretical physics. Recently they have appeared in diverse areas such as complexity theory, signal processing, numerical PDE, and quantum information theory. I have done foundational work to establish sound mathematical footing for these applications, including ground-breaking work in the study of rank and border rank of tensors, properties that arise in all the above-mentioned applications. The text [29] has become the standard reference for the area.

- Sginificant advancement in characterization of tensors of minimal border rank (with Michalek) [43].
- Gave first examples of explicit tensors with rank greater than twice their border rank (with Michalek), giving a counter-example to a conjecture of Rhodes, [43].
- Series of papers determining equations for secant varieties of Segre varieties (i.e. algebraic tests for border rank of tensors), and more generally of homogenous varieties [10, 40, 41, 51, 50, 10] (with various co-authors among Hauenstein, Ikenmeyer, Manivel, Ottaviani, Weyman)
- First explicit $m \times m \times m$ tensors of border rank at least $2m - 1$ [32].
- Major breakthrough in the study of ranks of polynomials by exploiting the singularities of corresponding hypersurfaces [48] (with Teitler), which has inspired a significant body of work on the subject.
- Answered question of Grasedyk on tensor network states, showing a widely used assumption in solid state physics was false [55] (with Qi and Ye).
- Established basic facts about tensor rank and border rank and more generally secant varieties of cominuscule varieties (with Buczynski) [3], [4].

Homogeneous varieties and representation theory. These works deal with the study of homogeneous varieties from the novel perspective of local projective differential geometry. They reveal remarkable algebraic structures that have been used by geometers in Gromov-Witten theory, Schubert calculus and other areas. Moreover they show how a considerable amount of representation theory can be deduced simply from the local differential geometry of homogenous varieties (e.g., the classification of complex simple Lie algebras [39]). In another direction, the project established significant new information about the exceptional groups and their representation theory, for example answering questions on categorical generalizations the exceptional series of Deligne in [34] and of Vogel in [37].

- Series of papers examining the projective differential geometry of homogeneous varieties (with Manivel), including a new proof of the classification of complex simple Lie algebras based on their projective differential geometry, and discovery of $E_{7\frac{1}{2}}$ [36, 33, 35, 42, 39].
- Series of papers investigating the geometry of homogeneous varieties associated to the exceptional groups, answering questions on categorical generalizations the exceptional series of Deligne in [34] and of Vogel in [37] (both with Manivel) and [42] (with Manivel and Westbury).
- Established surprising connections between longstanding conjectures in combinatorics (Alon-Tarsi) and algebraic geometry/representation theory (Hadamard-Howe), as well as certain integrals over unitary groups [16] (with Kumar).
- Series of papers studying Griffiths-Harris rigidity of homogeneous varieties, including a proof of their rigidity conjecture and establishment of new methods for EDS, with the introduction of Lie algebra cohomology techniques and filtered EDS. [57, 56] (with Robles), and [27, 24].

Projective geometry. These papers address classically flavored questions in projective algebraic geometry approached with modern methods from exterior differential systems and representation theory. Highlights include an explicit counter-example to the infinitesimal LeBrun-Salamon conjecture [59] on quaternionic-Kähler manifolds and contact Fano manifolds [38], affirmation of Hartshorne's conjecture on complete intersections for varieties cut out by quadrics [20], resolution of a conjecture of Kontsevich originating in physics calculations by translating it to classical algebraic geometry [23], counter-examples to a conjecture of Eisenbud-Koh-Stillmann [2], and resolution of the classical problem to determine defining equations for the variety of hypersurfaces with degenerate dual varieties [53].

- Counter-example to a conjecture of Eisenbud-Koh-Stillmann, and answer to a question of Eisenbud, on secant varieties of Veronese re-embeddings of varieties, [2] (with Buczynski and Ginesky).
- Explicit counter-example to the infinitesimal LeBrun-Salamon conjecture, namely construction of a smooth inhomogeneous Legendrian variety in dimension two. [38] (with Manivel).
- Geometrization and resolution of a conjecture of Kontsevich regarding Hadamard inverses of orthogonal matrices [23].
- Series of papers examining the existence and non-existence of linear spaces on projective varieties. Results include an $n!$ upper bound on the number of lines through a general point of a uni-ruled n-fold [25] and a differential-geometric test for uni-ruled-ness [22] that was used in [9] in their breakthrough on the Erdös distinct distances problem, a Fubini-type theorem in codimension two [46] (with Robles), and progress on the Debarre-deJong conjecture [49] (with Tommasi).
- Affirmation of Harshorne's conjecture for varieties cut out by quadrics and differential-geometric characterization of complete intersections [20].
- Established a new upper bound for the size of spaces of symmetric matrices of constant rank and its relation to degeneracy loci and dual varieties [12] (with Ilic).
- Differential-geometric proof of Zak's theorem on Severi varieties as well as establishment of infinitesimal structure of varieties with degenerate secant and tangential varieties [21].
- Determination of defining equations for the variety of hypersurfaces with degenerate dual varieties [53] (with Manivel and Ressayre).
- Singularities of varieties of small codimension can be detected just from second order projective differential geometric information at a general point [19].

Minimal submanifolds. My early work used methods from exterior differential systems and representation theory to work on classical questions about minimal submanifolds and generalizations of calibrated geometries.

- Generalization of calibrated geometry, Weierstrass type formulas for minimal 3-folds in 5-space, [18, 17].
- Determined existence and non-existence of minimal isometric embeddings of quasi-curved manifolds [13] (with Ivey).

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Department of Mathematics, Texas A&M University
E-mail address: jml@math.tamu.edu