

## A SUMMARY OF MY RESEARCH (SEPTEMBER 2017)

J.M. LANDSBERG

My research interests are broad: Cartan style differential geometry, classically influenced algebraic geometry, the geometry of homogeneous varieties, categorical generalizations of Lie algebras, the geometry and application of tensors, and most recently, complexity theory. I am the author of the graduate texts *Geometry and Complexity Theory* [57], *Tensors: geometry and applications* [33], and the co-author (with Ivey) of *Cartan for beginners* [17] (second edition [18]). What follows is a summary of results organized by themes.

**Complexity theory.** Algebraic complexity theory deals with determining the computational cost of common operations such as multiplying matrices or computing the discrete Fourier transform. There are three aspects: *lower bounds* – showing there does not exist a better algorithm, *upper bounds* – showing there does, and *algorithms*– finding fast algorithms. In 1969 Strassen discovered the standard algorithm for multiplying  $n \times n$  matrices is not optimal, and since then computer scientists have conjectured that as  $n$  goes to infinity, it becomes almost as easy to multiply two  $n \times n$  matrices as it is to add them. For the central problem of determining the complexity of matrix multiplication, there had been no progress on lower bounds for 25 years until my breakthrough papers [30, 60]. In current work, I am initiating a project to use algebraic geometry to prove upper bounds and find practical algorithms [52, 2]. A second central problem is permanent v. determinant, Valiant’s algebraic analog of  $\mathbf{P}$  v.  $\mathbf{NP}$ . I have several important results regarding permanent v. determinant, including the first exponential separation of permanent from determinant in any restricted model [58].

- First exponential separation of permanent from determinant in any restricted model [58] (with Ressayre).
- First improvement on the lower bound of the border rank the matrix multiplication tensor since 1985 [60] (with Ottaviani), as well as current world record [47] (with Michalek).
- Solved problem from 1980 to determine the border rank of the  $2 \times 2$  matrix multiplication tensor [30]. Gave second proof using completely different methods (with Hauenstein and Ikenmeyer) [13].
- First improvement in the lower bound of the rank the matrix multiplication tensor since 1999 [34]. Current world record was obtained by students working under my direction [67].
- Establishment of mathematical foundations of Geometric Complexity Theory, an approach to Valiant’s conjecture via algebraic geometry and representation theory proposed in [68, 69] by Mulmuley and Sohoni [6] (with Burgisser, Manivel and Weyman) and [35].
- Current world record for the GCT separation of permanent from determinant [59] (with Manivel and Ressayre).
- Proved the celebrated method of shifted partial derivatives, despite hopes of computer scientists, cannot be used to prove any significant separation of permanent from determinant [9] (with Efremenko, Schenck, and Weyman) and [11] (with Gesmundo).
- Geometric study of algorithms of Smirnov and Bini et. al. [52] (with Ryder).

- New decompositions of the  $3 \times 3$  matrix multiplication tensor found with the aid of geometry [2] (with Ballard, Ikenmeyer and Ryder).
- Extremal characterizations of the Coppersmith-Winograd tensor used in [7, 64] to prove the best upper bounds for the exponent of matrix multiplication that indicate it will be difficult to find a better tensor for the Coppersmith-Winograd method as proposed in [1], [48, 49] (with Michalek).
- Establishment of methods from algebraic geometry to prove matrix rigidity, including the first non-classical equations for non-rigidity [10] (with Gesmundo, Hauenstein, and Ikenmeyer).
- Established conditions for modules of obstructions in the sense of GCT to be useful, which was used by Ikenmeyer-Panova [14] to disprove the central conjecture of GCT [19] (with Kadish).
- Determined geometric formulation via spinors, and generalizations of, Valiant's holographic algorithms [50] (with Morton and Norine) and [32].

**Tensors, Waring problems, and applied tensor questions.** Tensors have long been used in theoretical physics. Recently they have appeared in diverse areas such as complexity theory, signal processing, numerical PDE, and quantum information theory. I have done foundational work to establish sound mathematical footing for these applications, including ground-breaking work in the study of rank and border rank of tensors, properties that arise in all the above-mentioned applications. Thanks to text [33], I have been recruited for geometric assistance with tensor problems in diverse application areas [66, 8].

- World record for symmetric border rank lower bounds of explicit polynomials [11] (with Gesmundo).
- Significant advancement in characterization of tensors of minimal border rank (with Michalek) [48].
- First examples of explicit tensors with rank greater than twice their border rank, giving a counter-example to a conjecture of Rhodes [48] (with Michalek).
- Series of papers determining equations for secant varieties of Segre varieties (i.e., algebraic tests for border rank of tensors), and more generally of homogenous varieties [13, 44, 45, 56, 55, 13] (with various co-authors among Hauenstein, Ikenmeyer, Manivel, Ottaviani, Weyman)
- First explicit  $m \times m \times m$  tensors of border rank at least  $2m - 1$  [36].
- Major breakthrough in the study of ranks of polynomials by exploiting the singularities of corresponding hypersurfaces [53] (with Teitler), which has inspired a significant body of work on the subject.
- Answered question of Grasedyk on tensor network states, showing a widely used assumption in solid state physics was false [61] (with Qi and Ye).
- Established basic facts about tensor rank and border rank and more generally secant varieties of cominuscule varieties [4, 5] (with Buczynski).

**Homogeneous varieties and representation theory.** My work in this area deals with the study of homogeneous varieties from the novel perspective of local projective differential geometry. The work has revealed remarkable algebraic structures that have been used by geometers in Gromov-Witten theory, Schubert calculus and other areas. Moreover it shows how a considerable amount of representation theory can be deduced simply from the local differential geometry of homogenous varieties (e.g., the classification of complex simple Lie algebras [43]). In another direction, the project established significant new information about the exceptional groups and their representation theory, for example answering questions on categorical generalizations of the

exceptional series due to Deligne [38] and categorical generalizations of general simple (super) Lie algebras due to Vogel [41].

- Series of papers examining the projective differential geometry of homogeneous varieties including a new proof of the classification of complex simple Lie algebras based on their projective differential geometry, and discovery of  $E_{7\frac{1}{2}}$  [40, 37, 39, 46, 43] (with Manivel).
- Series of papers investigating the geometry of homogeneous varieties associated to the exceptional groups, answering questions on categorical generalizations the exceptional series due to Deligne [38] and of general simple (super) Lie algebras due to Vogel [41] (both with Manivel) and [46] (with Manivel and Westbury).
- Established surprising connections between longstanding conjectures in combinatorics (Alon-Tarsi) and algebraic geometry/representation theory (Hadamard-Howe), as well as certain integrals over unitary groups [20] (with Kumar).
- Series of papers studying Griffiths-Harris rigidity of homogeneous varieties, including a proof of their rigidity conjecture and establishment of new methods for exterior differential systems (EDS), with the introduction of Lie algebra cohomology techniques and filtered EDS. [63, 62] (with Robles), and [31, 28].

**Projective geometry.** My work in this area addresses classically flavored questions in projective algebraic geometry approached with modern methods from exterior differential systems and representation theory. Highlights include an explicit counter-example to the infinitesimal LeBrun-Salamon conjecture [65] on quaternionic-Kähler manifolds and contact Fano manifolds [42], affirmation of Hartshorne’s conjecture on complete intersections for varieties cut out by quadrics [24], resolution of a conjecture of Kontsevich originating in physics calculations by translating it to classical algebraic geometry [27], counter-examples to a conjecture of Eisenbud-Koh-Stillmann [3], and resolution of the classical problem to determine defining equations for the variety of hypersurfaces with degenerate dual varieties [59].

- Counter-example to a conjecture of Eisenbud-Koh-Stillmann, and answer to a question of Eisenbud, on secant varieties of Veronese re-embeddings of varieties, [3] (with Buczyński and Giesky).
- Explicit counter-example to the infinitesimal LeBrun-Salamon conjecture, namely construction of a smooth inhomogeneous Legendrian variety in dimension two [42] (with Manivel).
- Geometrization and resolution of a conjecture of Kontsevich regarding Hadamard inverses of orthogonal matrices [27].
- Series of papers examining the existence and non-existence of linear spaces on projective varieties. Results include an  $n!$  upper bound on the number of lines through a general point of a uni-ruled  $n$ -fold [29] and a differential-geometric test for uni-ruledness [26] that was used in [12] in their breakthrough on the Erdős distinct distances problem, a Fubini-type theorem in codimension two [51] (with Robles), and progress on the Debarre-deJong conjecture [54] (with Tommasi).
- Affirmation of Hartshorne’s conjecture for varieties cut out by quadrics and differential-geometric characterization of complete intersections [24].
- Established a new upper bound for the size of spaces of symmetric matrices of constant rank and its relation to degeneracy loci and dual varieties [15] (with Ilic).
- Differential-geometric proof of Zak’s theorem on Severi varieties as well as establishment of infinitesimal structure of varieties with degenerate secant and tangential varieties [25].
- Determination of defining equations for the variety of hypersurfaces with degenerate dual varieties [59] (with Manivel and Ressayre).

- Singularities of varieties of small codimension can be detected just from second order projective differential geometric information at a general point [23].

**Minimal submanifolds.** My early work used methods from exterior differential systems and representation theory to work on classical questions about minimal submanifolds and generalizations of calibrated geometries.

- Generalization of calibrated geometry, Weierstrass type formulas for minimal 3-folds in 5-space, [22, 21].
- Determined existence and non-existence of minimal isometric embeddings of quasi-curved manifolds [16] (with Ivey).

#### REFERENCES

1. Andris Ambainis, Yuval Filmus, and François Le Gall, *Fast matrix multiplication: limitations of the Coppersmith-Winograd method (extended abstract)*, STOC'15—Proceedings of the 2015 ACM Symposium on Theory of Computing, ACM, New York, 2015, pp. 585–593. MR 3388238
2. Grey Ballard, Christian Ikenmeyer, J.M. Landsberg, and Nick Ryder, *The geometry of rank decompositions of matrix multiplication ii: 3x3 matrices*, preprint.
3. Jaroslaw Buczyński, Adam Ginensky, and J. M. Landsberg, *Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture*, J. Lond. Math. Soc. (2) **88** (2013), no. 1, 1–24. MR 3092255
4. Jaroslaw Buczyński and J. M. Landsberg, *Ranks of tensors and a generalization of secant varieties*, Linear Algebra Appl. **438** (2013), no. 2, 668–689. MR 2996361
5. ———, *On the third secant variety*, J. Algebraic Combin. **40** (2014), no. 2, 475–502. MR 3239293
6. Peter Bürgisser, J. M. Landsberg, Laurent Manivel, and Jerzy Weyman, *An overview of mathematical issues arising in the geometric complexity theory approach to  $VP \neq VNP$* , SIAM J. Comput. **40** (2011), no. 4, 1179–1209. MR 2861717
7. Don Coppersmith and Shmuel Winograd, *Matrix multiplication via arithmetic progressions*, J. Symbolic Comput. **9** (1990), no. 3, 251–280. MR 91i:68058
8. Fernando De Terán, Froilán M. Dopico, and J. M. Landsberg, *An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank*, Linear Algebra Appl. **520** (2017), 80–103. MR 3611459
9. K. Efremenko, J. M. Landsberg, H. Schenck, and J. Weyman, *On minimal free resolutions and the method of shifted partial derivatives in complexity theory*, ArXiv e-prints (2015).
10. Fulvio Gesmundo, Jonathan D. Hauenstein, Christian Ikenmeyer, and J. M. Landsberg, *Complexity of linear circuits and geometry*, Found. Comput. Math. **16** (2016), no. 3, 599–635. MR 3494506
11. Fulvio Gesmundo and JM Landsberg, *Explicit polynomial sequences with maximal spaces of partial derivatives and a question of k. mulmuley*, arXiv:1705.03866.
12. Larry Guth and Nets Hawk Katz, *On the Erdős distinct distances problem in the plane*, Ann. of Math. (2) **181** (2015), no. 1, 155–190. MR 3272924
13. Jonathan D. Hauenstein, Christian Ikenmeyer, and J. M. Landsberg, *Equations for lower bounds on border rank*, Exp. Math. **22** (2013), no. 4, 372–383. MR 3171099
14. C. Ikenmeyer and G. Panova, *Rectangular Kronecker coefficients and plethysms in geometric complexity theory*, ArXiv e-prints (2015).
15. Bo Ilic and J. M. Landsberg, *On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties*, Math. Ann. **314** (1999), no. 1, 159–174. MR MR1689267 (2000e:14091)
16. Thomas Ivey and J. M. Landsberg, *On isometric and minimal isometric embeddings*, Duke Math. J. **89** (1997), no. 3, 555–576. MR 1470342 (98g:53109)
17. Thomas A. Ivey and J. M. Landsberg, *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, RI, 2003. MR 2003610 (2004g:53002)
18. Thomas A. Ivey and Joseph M. Landsberg, *Cartan for beginners*, Graduate Studies in Mathematics, vol. 175, American Mathematical Society, Providence, RI, 2016, Differential geometry via moving frames and exterior differential systems, Second edition [of MR2003610]. MR 3586335
19. Harlan Kadish and J. M. Landsberg, *Padded polynomials, their cousins, and geometric complexity theory*, Comm. Algebra **42** (2014), no. 5, 2171–2180. MR 3169697

20. Shrawan Kumar and J. M. Landsberg, *Connections between conjectures of Alon-Tarsi, Hadamard-Howe, and integrals over the special unitary group*, Discrete Math. **338** (2015), no. 7, 1232–1238. MR 3322811
21. J. M. Landsberg, *Minimal submanifolds of  $E^{2n+1}$  arising from degenerate  $SO(3)$  orbits on the Grassmannian*, Trans. Amer. Math. Soc. **325** (1991), no. 1, 101–117. MR 1012515 (91h:53055)
22. ———, *Minimal submanifolds defined by first-order systems of PDE*, J. Differential Geom. **36** (1992), no. 2, 369–415. MR 1180388 (93k:58058)
23. ———, *On second fundamental forms of projective varieties*, Invent. Math. **117** (1994), no. 2, 303–315. MR 1273267 (95g:14057)
24. ———, *Differential-geometric characterizations of complete intersections*, J. Differential Geom. **44** (1996), no. 1, 32–73. MR 1420349 (98d:14062)
25. ———, *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. J. **85** (1996), no. 3, 605–634. MR 1422359 (98d:14066)
26. ———, *Is a linear space contained in a submanifold? On the number of derivatives needed to tell*, J. Reine Angew. Math. **508** (1999), 53–60. MR 1676869 (2000k:14035)
27. ———, *On a conjecture of Kontsevich and variants of Castelnuovo’s lemma*, Compositio Math. **115** (1999), no. 2, 231–239. MR 1668998
28. ———, *On the infinitesimal rigidity of homogeneous varieties*, Compositio Math. **118** (1999), no. 2, 189–201. MR 1713310 (2000j:14082)
29. ———, *Lines on projective varieties*, J. Reine Angew. Math. **562** (2003), 1–3. MR 2011327 (2004i:14063)
30. ———, *The border rank of the multiplication of  $2 \times 2$  matrices is seven*, J. Amer. Math. Soc. **19** (2006), no. 2, 447–459. MR 2188132 (2006j:68034)
31. ———, *Griffiths-Harris rigidity of compact Hermitian symmetric spaces*, J. Differential Geom. **74** (2006), no. 3, 395–405. MR 2269783 (2007m:32014)
32. ———, *P versus NP and geometry*, J. Symbolic Comput. **45** (2010), no. 12, 1369–1377. MR 2733384 (2012c:68065)
33. ———, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. MR 2865915
34. ———, *New lower bounds for the rank of matrix multiplication*, SIAM J. Comput. **43** (2014), no. 1, 144–149. MR 3162411
35. ———, *Geometric complexity theory: an introduction for geometers*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **61** (2015), no. 1, 65–117. MR 3343444
36. ———, *Nontriviality of equations and explicit tensors in  $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$  of border rank at least  $2m - 2$* , J. Pure Appl. Algebra **219** (2015), no. 8, 3677–3684. MR 3320240
37. J. M. Landsberg and L. Manivel, *The projective geometry of Freudenthal’s magic square*, J. Algebra **239** (2001), no. 2, 477–512. MR 1832903 (2002g:14070)
38. ———, *Triality, exceptional Lie algebras and Deligne dimension formulas*, Adv. Math. **171** (2002), no. 1, 59–85. MR 1933384 (2003i:17012)
39. ———, *Series of Lie groups*, Michigan Math. J. **52** (2004), no. 2, 453–479. MR 2069810 (2005f:17009)
40. ———, *The sextonions and  $E_{7\frac{1}{2}}$* , Adv. Math. **201** (2006), no. 1, 143–179. MR 2204753 (2006k:17017)
41. ———, *A universal dimension formula for complex simple Lie algebras*, Adv. Math. **201** (2006), no. 2, 379–407. MR 2211533 (2006k:17011)
42. ———, *Legendrian varieties*, Asian J. Math. **11** (2007), no. 3, 341–359. MR 2372722 (2008k:14082)
43. J. M. Landsberg and Laurent Manivel, *Construction and classification of complex simple Lie algebras via projective geometry*, Selecta Math. (N.S.) **8** (2002), no. 1, 137–159. MR 1 890 196
44. ———, *On the ideals of secant varieties of Segre varieties*, Found. Comput. Math. **4** (2004), no. 4, 397–422. MR MR2097214 (2005m:14101)
45. ———, *Generalizations of Strassen’s equations for secant varieties of Segre varieties*, Comm. Algebra **36** (2008), no. 2, 405–422. MR MR2387532
46. J. M. Landsberg, Laurent Manivel, and Bruce W. Westbury, *Series of nilpotent orbits*, Experiment. Math. **13** (2004), no. 1, 13–29. MR 2065565 (2005e:17018)
47. J. M. Landsberg and M. Michalek, *On the geometry of border rank algorithms for matrix multiplication and other tensors with symmetry*, ArXiv e-prints (2016).
48. J. M. Landsberg and Mateusz Michalek, *Abelian tensors*, J. Math. Pures Appl. (9) **108** (2017), no. 3, 333–371. MR 3682743
49. ———, *On the geometry of border rank decompositions for matrix multiplication and other tensors with symmetry*, SIAM J. Appl. Algebra Geom. **1** (2017), no. 1, 2–19. MR 3633766

50. J. M. Landsberg, Jason Morton, and Serguei Norine, *Holographic algorithms without matchgates*, Linear Algebra Appl. **438** (2013), no. 2, 782–795. MR 2996367
51. J. M. Landsberg and Colleen Robles, *Fubini’s theorem in codimension two*, J. Reine Angew. Math. **631** (2009), 221–235. MR 2542223 (2010h:14078)
52. J. M. Landsberg and Nicholas Ryder, *On the Geometry of Border Rank Algorithms for  $n \times 2$  by  $2 \times 2$  Matrix Multiplication*, Exp. Math. **26** (2017), no. 3, 275–286. MR 3642105
53. J. M. Landsberg and Zach Teitler, *On the ranks and border ranks of symmetric tensors*, Found. Comput. Math. **10** (2010), no. 3, 339–366. MR 2628829 (2011d:14095)
54. J. M. Landsberg and Orsola Tommasi, *On the Debarre-de Jong and Beheshti-Starr conjectures on hypersurfaces with too many lines*, Michigan Math. J. **59** (2010), no. 3, 573–588. MR 2745753 (2012a:14121)
55. J. M. Landsberg and Jerzy Weyman, *On the ideals and singularities of secant varieties of Segre varieties*, Bull. Lond. Math. Soc. **39** (2007), no. 4, 685–697. MR MR2346950
56. ———, *On secant varieties of compact Hermitian symmetric spaces*, J. Pure Appl. Algebra **213** (2009), no. 11, 2075–2086. MR 2533306 (2010i:14095)
57. J.M. Landsberg, *Geometry and complexity theory*, preprint available at <http://www.math.tamu.edu/~jml/simonsclass.pdf>.
58. J.M. Landsberg and Nicolas Ressayre, *Permanent v. determinant: an exponential lower bound assuming symmetry and a potential path towards valiant’s conjecture*, DGA special issue (2017).
59. Joseph M. Landsberg, Laurent Manivel, and Nicolas Ressayre, *Hypersurfaces with degenerate duals and the geometric complexity theory program*, Comment. Math. Helv. **88** (2013), no. 2, 469–484. MR 3048194
60. Joseph M. Landsberg and Giorgio Ottaviani, *New lower bounds for the border rank of matrix multiplication*, Theory Comput. **11** (2015), 285–298. MR 3376667
61. Joseph M. Landsberg, Yang Qi, and Ke Ye, *On the geometry of tensor network states*, Quantum Inf. Comput. **12** (2012), no. 3-4, 346–354. MR 2933533
62. Joseph M. Landsberg and Colleen Robles, *Fubini-Griffiths-Harris rigidity and Lie algebra cohomology*, Asian J. Math. **16** (2012), no. 4, 561–586. MR 3004278
63. ———, *Fubini-Griffiths-Harris rigidity of homogeneous varieties*, Int. Math. Res. Not. IMRN (2013), no. 7, 1643–1664. MR 3044453
64. Francois Le Gall, *Powers of tensors and fast matrix multiplication*, arXiv:1401.7714.
65. Claude LeBrun and Simon Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118** (1994), no. 1, 109–132. MR 95k:53059
66. F. Malgouyres and J. Landsberg, *Stable recovery of the factors from a deep matrix product and application to convolutional network*, ArXiv e-prints (2017).
67. Alex Massarenti and Emanuele Raviolo, *The rank of  $n \times n$  matrix multiplication is at least  $3n^2 - 2\sqrt{2}n^{\frac{3}{2}} - 3n$* , Linear Algebra Appl. **438** (2013), no. 11, 4500–4509. MR 3034546
68. Ketan D. Mulmuley and Milind Sohoni, *Geometric complexity theory. I. An approach to the P vs. NP and related problems*, SIAM J. Comput. **31** (2001), no. 2, 496–526 (electronic). MR MR1861288 (2003a:68047)
69. ———, *Geometric complexity theory. II. Towards explicit obstructions for embeddings among class varieties*, SIAM J. Comput. **38** (2008), no. 3, 1175–1206. MR MR2421083

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY

E-mail address: [jml@math.tamu.edu](mailto:jml@math.tamu.edu)