

**The Geometry of Tensors with
applications
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please send
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Preface

Tensors are ubiquitous in the sciences. They provide a useful way to organize data. The geometry describing qualitative properties of tensors is a powerful tool for extracting information from data sets and a beautiful subject in its own right. This book has three intended uses: as a classroom textbook, a reference work for researchers, and a research manuscript.

Classroom uses. The first part of this text is suitable for a course in multi-linear algebra - it provides a solid foundation for the study of tensors and contains numerous applications, exercises, and examples. The book is also suitable for advanced courses. In preliminary form, it was used by the author to give graduate classes on the complexity of matrix multiplication, algebraic complexity theory, and the geometry of secant varieties (see §0.2.5 below for more detail).

Reference uses. I have compiled information on tensors in table format (e.g. regarding border rank, maximal rank, typical rank etc..) for easy reference. Up until now there had been no reference for even the classical results regarding tensors. I have also included a dictionary at the end of Chapter 2 to relate standard mathematical terminology to the tensor literature (e.g., [103, 50]). (Caveat: I do not include results relying on a metric or Hermitian metric.)

Research uses. I have tried to state all the results and definitions from geometry and representation theory needed to study tensors. When proofs are not included, precise references for them are given. The text includes the state of the art regarding ranks and border ranks of tensors, and explains for the first time many results and problems coming from outside mathematics in geometric language. For example, a very short proof of the well-known Kruskal theorem is presented, illustrating that it hinges upon a basic geometric fact about point sets in projective space. Numerous open problems are presented throughout the text.

0.1. Overview

Chapter 1: Motivating problems. I present problems from complexity theory (complexity of matrix multiplication), statistics (phylogenetic invariants), and signal processing (blind source separation) and show how they all lead to the study of *secant varieties of Segre and Veronese varieties*. The motivating questions all deal with subsets (usually algebraic subvarieties) of spaces of tensors invariant under group actions.

This chapter should be accessible to anyone who is scientifically literate. It is motivational rather than systematic.

Chapter 2: Multilinear algebra. The purpose of this chapter is to introduce the language of *tensors*. While non-mathematicians often think of tensors as n -dimensional $\mathbf{a}_1 \times \cdots \times \mathbf{a}_n$ -tables, I emphasize coordinate free definitions. The coordinate free descriptions make it easier for one to take advantage of symmetries and to apply theorems. Chapter 2 includes: numerous exercises where familiar notions from linear algebra are presented in an invariant context, a discussion of rank and border rank, invariant descriptions of several polynomials arising naturally in graph theory, and first steps towards explaining how to decompose spaces of tensors. Two appendices are included. The first translates between tensor language and geometric language, which should aid tensorists in reading the relevant literature in geometry. The second describes *wiring diagrams*, a pictorial tool for understanding the invariance properties of tensors and as a tool for aiding calculations.

This chapter should be accessible to anyone who has had a first course in linear algebra. It may be used as the basis of a course in multi-linear algebra.

Chapter 3: Projective algebraic geometry. A central task to be accomplished in many of the motivating problems is to test if a given tensor has membership in a given set (e.g., if a given tensor has *rank* r). Some of these sets are defined as the zero sets of collections of polynomials, i.e., as *algebraic varieties*, while others can be expanded to be varieties by taking their Zariski closure (e.g. the set of tensors of border rank at most r is the Zariski closure of the set of tensors of rank at most r). I present only the essentials of projective geometry here, in order to quickly arrive at the study of groups and G -modules essential to this book. I include a discussion of secant varieties as they play a central role in applications. Other topics in algebraic geometry are introduced as needed.

This chapter may be difficult for those unfamiliar with algebraic geometry - it is terse as numerous excellent references are available (e.g. [79, 166]). Its purpose is primarily to establish language and it may be skimmed on a first reading.

Chapter 4: Exploiting symmetry: Representation theory for spaces of tensors. Representation theory provides a language for taking advantage of symmetries. Consider the space $Mat_{n \times m}$ of $n \times m$ matrices: one is usually interested in the properties of a matrix up to changes of bases (that is, the underlying properties of the linear map it encodes). This is an example of a vector space with a group acting on it. Consider polynomials on the space of matrices. The minors are the most important polynomials. But now consider the space of k -way arrays (i.e., a space of tensors) with

$k > 2$. What are the spaces of important polynomials? Representation theory helps to give an answer.

Chapter 4 covers representations of the group of permutations on d elements, denoted \mathfrak{S}_d , the group of invertible $n \times n$ matrices, denoted $GL_n\mathbb{C}$, Schur duality, and the decomposition of $V^{\otimes d}$ as a $GL(V)$ -module.

The material presented in this chapter is standard and excellent texts already exist (e.g., [159, 69, 74]). I focus on the aspects of representation theory useful for applications and its implementation.

The prerequisite for this chapter is Chapter 2.

Chapter 5: First examples of G -varieties and their equations. The ideals of the varieties of rank one tensors and symmetric tensors (as well as of some other homogeneous varieties) are described in two different ways - using representation theory, and using the interpretation of the varieties as the closure of the space of vectors of minors. The latter interpretation is used in the study of holographic algorithms. Several other varieties relevant for applications are discussed and their defining equations are derived.

The prerequisites for this chapter are Chapters 3 and 4.

Chapter 6: Tests for border rank: equations for secant varieties. This chapter discusses secant varieties in general and the equations of secant varieties of the varieties of rank one tensors and symmetric tensors, i.e., the varieties of tensors, and symmetric tensors of border rank at most r . These are the most important objects for tensor decomposition so an effort is made to present the state of the art and to give as many different perspectives as possible.

Chapter 6's prerequisite is Chapter 5.

Chapter 7: Additional varieties useful for spaces of tensors. In addition to the varieties governing border rank, there are other natural varieties occurring in spaces of tensors that play a role in classifying normal forms and the study of rank. They also should be useful for future applications. As with secant varieties, they are best described in a more general geometric setting. This chapter focuses on two such: dual varieties and tangential varieties. Dual varieties play a role in distinguishing the different *typical ranks* that can occur for tensors over the real numbers. Differential-geometric tools for studying these varieties are also presented here.

Chapter 7 can be read immediately after Chapter 3.

Chapter 8: Uniqueness of tensor decompositions. Often in applications one would like unique expressions for tensors as a sum of decomposable tensors. This is often not possible and the first topic of the chapter is to

bring the reader up to date on what is known regarding when a unique expression is possible. Next the often cited Kruskal uniqueness condition for tensors is stated, and a geometric proof of the theorem is given. This should be particularly interesting for its brevity and because it isolates the basic geometric statement that underlies the result.

The chapter can be read on a basic level after reading Chapter 2, but the proofs require Chapters 3,4 and 7.

Chapter 9: Normal forms for small tensors. The chapter describes the spaces of tensors admitting normal forms, and the normal forms of tensors in those spaces, as well as normal forms for points in small secant varieties.

The chapter can be read on a basic level after reading Chapter 2, but the proofs and geometric descriptions of the various orbit closures require Chapters 3,4 and 7.

Chapter 10: Rank. It is more natural in algebraic geometry to discuss border rank than rank because it relates to projective varieties, yet for applications sometimes one needs to study the rank of tensors and symmetric tensors. I first regard rank in a more general geometric context, and then specialize to the cases of interest for applications. Very little is known about the possible ranks of tensors, and what little is known is mostly in cases where there are normal forms, which was presented in Chapter 8. The main discussion in this chapter regards the ranks of symmetric tensors. Included are the Comas-Seguir theorem classifying ranks of symmetric tensors in two variables as well as recent results.

The results of Chapter 10 are easily understood after skimming Chapter 3, however the proofs use results from algebraic geometry that may be unfamiliar to some readers. (However the results used, such as Bertini's Theorem and the Lefschetz hyperplane Theorem, are standard and their statements are given in the chapter).

Chapter 11: Conjectures of Comon, Eisenbud and Strassen. This chapter introduces a geometric notion: that of a *(border) rank preserving pair*, as it gives common language to conjectures in signal processing, algebraic geometry and complexity theory. *It will appear in the final version of the book but is currently being written.* An important application to the complexity of matrix multiplication is described, namely a geometric explanation of Schönhage's approximate algorithm for multiplying 3×3 matrices using 21 multiplications.

The next three chapters deal with applications.

Chapter 12: The complexity of matrix multiplication. This chapter brings the reader up to date on what is known regarding the complexity of matrix multiplication, including new proofs of many standard results. *In this version of the book the chapter is incomplete as I am working on better proofs for some of the existing results.*

Much of the chapter needs only Chapter 2, but parts require results from Chapters 3,4 and 11.

Chapter 13: P versus NP. This chapter includes an introduction to several algebraic versions of **P** and **NP**, as well as a discussion of Valiant's *holographic algorithms*.

It has §5.5 as a pre-requisite.

This chapter will also include a discussion of the GCT program of Mulmuley and Sohoni in a later version.

Chapter 14: Applications to statistics and signal processing. *This chapter has yet to be written. I plan to include a discussion of cumulants as tensors and how tensor decomposition is used in several applications. While this area is not my expertise, several experts have offered to help me with the writing of this chapter.*

The final three chapters deal with more advanced topics.

Chapter 15: Outline of the proof of the Alexander-Hirschowitz theorem. The dimensions of the varieties of symmetric tensors of border rank at most r has been completely solved by Alexander and Hirschowitz. A brief outline of a streamlined proof appearing in [154] is given here.

This chapter is intended for someone who has already had a basic course in algebraic geometry.

Chapter 16: Representation theory. This chapter includes a brief description of the rudiments of the representation theory of complex simple Lie algebras. There are many excellent references for this subject so I present just enough of the theory for our purposes: the proof of Kostant's theorem that the ideals of homogeneous varieties are generated in degree two, the statement of the Bott-Borel-Weil theorem, and the presentation of the *inheritance* principle of Chapter 6 in a more general context.

This chapter is intended for someone who has already had a first course in representation theory.

Chapter 17: Weyman’s method. The study of secant varieties of triple Segre products naturally leads to the Kempf-Weyman method for determining ideals and singularities of G -varieties. This chapter contains an exposition of the rudiments of Weyman’s method, intended primarily to serve as an introduction to the book [195].

The prerequisites for this chapter include Chapter 16 and a first course in algebraic geometry.

0.2. Conventions, history, acknowledgments

0.2.1. Notations. V, A_j are finite dimensional complex vector spaces. If $v_1, \dots, v_p \in V$, $\langle v_1, \dots, v_p \rangle$ denotes the span of v_1, \dots, v_p . If e_1, \dots, e_n is a basis of V , e^1, \dots, e^n denotes the dual basis of the dual space V^* . $GL(V)$ denotes the general linear group of invertible linear maps $V \rightarrow V$ and $\mathfrak{gl}(V)$ its Lie algebra of endomorphisms of V . If G denotes a Lie or algebraic group, \mathfrak{g} denote its associated Lie algebra.

If $X \subset \mathbb{P}V$ an algebraic set, then $\hat{X} \subset V$ is the cone over it, its inverse image plus 0 under $\pi : V \setminus 0 \rightarrow \mathbb{P}V$. If $v \in V$, $[v] \in \mathbb{P}V$ denotes $\pi(v)$. The linear span of a set $X \subset \mathbb{P}V$ is denoted $\langle X \rangle \subseteq V$.

For a variety X , X_{smooth} denotes its smooth points and X_{sing} denotes its singular points.

Vector spaces are usually denoted A, B, C, V, W, A_j and the dimensions are usually the corresponding bold letters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ etc...

$\Lambda^k V$ denotes the k -th exterior power of the vector space V , the symbols \wedge and \bigwedge denote exterior product. $S^k V$ is the k -th symmetric power. The tensor product of $v, w \in V$ is denoted $v \otimes w \in V^{\otimes 2}$, and symmetric product has no marking, e.g., $vw = \frac{1}{2}(v \otimes w + w \otimes v)$. If $p \in S^d V$ is a homogeneous polynomial of degree d , write $p_{k,d-k} \in S^k V \otimes S^{d-k} V$ for its partial polarization and \bar{p} for p considered as a d -multilinear form $V^* \times \dots \times V^* \rightarrow \mathbb{C}$.

\mathfrak{S}_d denotes the group of permutations on d elements and $GL(V)$ denotes the group of invertible linear maps $V \rightarrow V$.

To a partition $\pi = (p_1, \dots, p_r)$ of d , i.e., a set of integers $p_1 \geq p_2 \geq \dots \geq p_r$, $p_i \in \mathbb{Z}_+$, such that $p_1 + \dots + p_r = d$, $[\pi]$ denotes the associated irreducible \mathfrak{S}_d module and $S_\pi V$ denotes the associated irreducible $GL(V)$ -module. I write $|\pi| = d$, and $\ell(\pi) = r$.

0.2.2. Layout. All theorems, propositions, remarks, examples, etc., are numbered together within each section; for example, Theorem 1.3.2 is the second numbered item in Section 1.3. Equations are numbered sequentially within each Chapter. I have included hints for selected exercises, those

marked with the symbol \odot at the end, which is meant to be suggestive of a life preserver.

0.2.3. Further reading. For gaining a basic grasp of representation theory as used in this book, one could consult [74, 69, 159, 89]. The styles of these books vary significantly and the reader's taste will determine which she or he prefers. To go further with representation theory [102] is useful, especially for the presentation of the Weyl character formula. An excellent (and pictorial!) presentation of the implementation of the Bott-Borel-Weil theorem is in [9].

For basic algebraic geometry as in Chapter 3, [79, 166] are useful. For the more advanced commutative algebra needed in the later chapters [63] is written with geometry in mind. The standard and only reference for Weyman's method is [195].

The standard reference for what was known in algebraic complexity theory up to 1997 is [24].

0.2.4. History. *This will be a short history of the uses of tensors*

0.2.5. Acknowledgments. This project started as a collaboration with Jason Morton, who contributed significantly to the writing and editing of chapters 2,3 and 4. The book has greatly benefitted from his input. The first draft of this book arose out of a series of lectures I was invited to give by B. Sturmfels for his working group at UC Berkeley in spring 2006. I then gave a graduate class at Texas A&M University in fall 2007 on the complexity of matrix multiplication and a class on complexity theory in spring 2009. J. Morton and I gave a summer graduate workshop at MSRI in July 2008, as well as several lectures at a follow-up research workshop at AIM in July 2008. I also gave a GNSAGA lecture series in Florence, Italy in June 2009 on secant varieties. It is a pleasure to thank the students in these classes at A&M and MSRI as well as my hosts Sturmfels, MSRI, AIM and Ottaviani. Much of the material in this book comes from joint work with L. Manivel, G. Ottaviani, and J. Weyman. It is a pleasure to thank these three collaborators for significant help at each step along the way. Other material comes from joint work with the post-docs J. Buczinski, J. Morton, S. Norine, and Z. Teitler, who have also helped with the book, and I am fortunate to be able to thank them as well. I also thank my students and post-docs A. Boralevi, L. Nguyen, L. Oeding, Y. Qi, M. Yang, and K. Ye for their comments and questions, as well as colleagues J-Y Cai, P. Comon, and L-H Lim, for patiently answering my questions. [*I am also scheduled to give a short course from this book June 2010 in Norway.*]