Math 323 Exam 1, 2/13/13

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You may use one page of handwritten notes - no photocopies, no calculators.

1. Let $A = \begin{pmatrix} 1 & 7 & 0 \\ 2 & 5 & 1 \\ 3 & 1 & 2 \end{pmatrix}$. Find elementary matrices E_1, E_2, E_3 such that $E_1E_2E_3A$ is upper triangular, compute $E_1E_2E_3A$ and $\det A$. Write your answers here

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{20}{9} & 1 \end{pmatrix}.$$

$$E_1E_2E_3A = \begin{pmatrix} 1 & 7 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{2}{9} \end{pmatrix}.$$

$\det A = 1 * (-9) * (-\frac{2}{9}) = 2.$

2. Let A, B be $n \times n$ matrices and let $C = AB$. Prove that if B is singular then C is singular.

Proof. First proof: C is singular iff $\det(C) = 0$, but $\det(C) = \det(AB) = \det(A)\det(B)$ so if $\det(B) = 0$, then $\det(C) = 0$.

Second proof: C is singular iff $C\mathbf{x} = 0$ fails to have a unique solution. If $B\mathbf{x} = 0$ fails to have a unique solution, e.g., say $B\mathbf{y} = 0$, then $C\mathbf{y} = AB\mathbf{y} = 0$ as well. $\square$

3. Solve the following system for $\mathbf{x} \in \mathbb{R}^4$. (Hint: think before you start working.)

$$\begin{pmatrix} 11 & 2 & 3 & 4 \\ 15 & 6 & 7 & 8 \\ 9 & 110 & 11 & 12 \\ 13 & 14 & 15 & 33 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 7 \\ 11 \\ 15 \end{pmatrix}$$

This is the system

$$x^1 \begin{pmatrix} 11 \\ 15 \\ 9 \\ 13 \end{pmatrix} + x^2 \begin{pmatrix} 2 \\ 6 \\ 110 \\ 14 \end{pmatrix} + x^3 \begin{pmatrix} 3 \\ 7 \\ 11 \\ 15 \end{pmatrix} + x^4 \begin{pmatrix} 4 \\ 8 \\ 12 \\ 33 \end{pmatrix} + = \begin{pmatrix} 3 \\ 7 \\ 11 \\ 15 \end{pmatrix}$$

so it clearly has the solution $x^1 = x^2 = x^4 = 0$, $x^3 = 1$.

4. Let

$$A = \begin{pmatrix} 1 & 7 & 0 & 4 \\ 2 & 5 & 1 & 8 \\ 3 & 1 & 2 & 12 \\ 13 & 14 & 15 & 33 \end{pmatrix}.$$

Given that $\det(A) = -38$, compute the value of x^4 in the solution of the equation

$$\begin{pmatrix} 1 & 7 & 0 & 4 \\ 2 & 5 & 1 & 8 \\ 3 & 1 & 2 & 12 \\ 13 & 14 & 15 & 33 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using Cramer's rule, we have

$$x^4 = \frac{\det \begin{pmatrix} 1 & 7 & 0 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 13 & 14 & 15 & 1 \end{pmatrix}}{\det A} = \frac{\det \begin{pmatrix} 1 & 7 & 0 \\ 2 & 5 & 1 \\ 3 & 1 & 2 \end{pmatrix}}{\det A}.$$

But we already calculated $\det \begin{pmatrix} 1 & 7 & 0 \\ 2 & 5 & 1 \\ 3 & 1 & 2 \end{pmatrix} = 2$, so $x^4 = \frac{2}{-38} = -\frac{1}{19}$.