Math 323 Exam 2 Answers, 3/6/14

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You may use up to 5 pages of handwritten notes - no photocopies, no calculators. For all questions on the test, no credit will be given without work/justification.

Special 3 point bonus question: What is your favorite radio station in town?

Landsberg's answer: KEOS 89.1 (For the class there was a wide range of answers with "candy 95" and "the fox" getting the most)

1. Let $C[0,1]$ denote the vector space of continuous functions on the unit interval, and $\mathbb{R}$ the one dimensional vector space. Which of the following maps $L : C[0,1] \to \mathbb{R}$ are linear? (Be sure to give proofs.)

   (a) $L(f) = \frac{1}{2}(f(\frac{3}{4}) - f(\frac{1}{4}))$.

   (b) $L(f) = \int_0^1 f(x)^2 dx$.

   Answer a) yes:

$$\begin{aligned} L(f_1 + cf_2) &= \frac{1}{2}((f_1 + cf_2)(\frac{3}{4}) - (f_1 + cf_2)(\frac{1}{4}) \\ &= \frac{1}{2}(f_1(\frac{3}{4}) + cf_2(\frac{3}{4}) - f_1(\frac{1}{4}) - cf_2(\frac{1}{4})) \\ &= \frac{1}{2}(f_1(\frac{3}{4}) - f_1(\frac{1}{4})) + c\frac{1}{2}(f_2(\frac{3}{4}) - f_2(\frac{1}{4})) \\ &= L(f_1) + cL(f_2) \end{aligned}$$

   b) no: Let $f(x)$ be the constant function $f(x) = 1$ and let $c = 2$. $L(cf) = \int_0^1 (2(1))^2 dx = \int_0^1 4dx = 4$, while $2L(f) = 2\int_0^1 1dx = 2$.

2. Let P_k denote the vector space of polynomials in one variable of degree at most $k-1$. Let $L : P_3 \to P_3$ be the linear map $f(x) \mapsto f'(x) - f(0)$.

   (a) Give P_3 the basis $v_1 = 1, v_2 = x, v_3 = x^2$ and write down the matrix representing L in this basis.

   Answer: Note that $L(v_1) = (-1)v_1 + 0v_2 + 0v_3$, $L(v_2) = (1)v_1 + 0v_2 + 0v_3$, $L(v_3) = 0v_1 + 2v_2 + 0v_3$ so the matrix is

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

   (b) What is the rank of L?

   Answer: Since the matrix is already in reduced form, we see the rank equals 2.

   (c) What is the kernel of L?

   Answer: The kernel is one dimensional by the previous part and the rank nullity theorem, and v_1+v_2 is clearly in the kernel so $\ker(L) = \text{span}\{v_1+v_2\}$.

3. Which of the following are spanning sets for P_4 (the polynomials of degree at most three)? (Be sure to furnish proofs.)

   (a) $\{v_1 = 1, v_2 = x, v_3 = x^2, v_4 = x^3, v_5 = 2x - 7\}$.

   (b) $\{v_1 = x, v_2 = x^2, v_3 = x^3, v_4 = 2x - 7\}$.

   (c) $\{v_1 = 1, v_2 = x, v_3 = x^3 - x^2\}$.

   Answers

   a. yes, already v_1, v_2, v_3, v_4 span (in fact they are the standard basis).

   b. yes. We can write $a + bx + cx^2 + dx^3 = -\frac{a}{7}v_4 + (b + \frac{2a}{7})v_1 + cv_2 + dv_3$.

   c. No, the dimension of the vector space is four, but only three vectors are given.

4. Find the transition matrix representing the change of basis on P_3 from the basis $E = \{v_1 = 1, v_2 = x, v_3 = x^2\}$ to the basis $F = \{w_1 = 1, w_2 = 1 + x, w_3 = 1 + x + x^2\}$.

   Answer: Since $w_1 = v_1$, $w_2 = v_1 + v_2$, $w_3 = v_1 + v_2 + v_3$, and the columns of the change of basis matrix are the coefficients of this expression, we obtain

$$S_F^E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   Since $v_1 = w_1$, $v_2 = w_2 - w_1$, $v_3 = w_3 - w_2$, and the columns of the change of basis matrix are the coefficients of this expression, we obtain

$$S_E^F = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$