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Math 323 Exam 3, 4/8/14 Answers

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1. Let P_k denote the vector space of polynomials of degree at most $k-1$. Define a linear map $L : P_4 \to P_3$ by $L(p(x)) = p'(x) + 3p''(x)$. Determine $\ker(L)$ and $Image(L)$.

   Answer: Every element of P_4 is of the form $p(x) = a + bx + cx^2 + dx^3$ for some constants a, b, c, d. We compute $L(p(x)) = (6c+b)1 + (18d+2c)x + (3d)x^2$. To have $L(p(x)) = \overline{0}$, i.e., $p(x) \in \ker L$, we need all the coefficients to be zero, which implies $b = c = d = 0$. Thus $\ker L = \{(a)1 \mid a \in \mathbb{R}\}$.

   By the rank nullity theorem $\dim(Image(L)) = \dim P_4 - \dim \ker(L) = 4 - 1 = 3$, but $\dim P_3 = 3$ so $Image(L) = P_3$.

2. Let $V = \text{span}\{v_1 := 1, v_2 := e^x, v_3 = e^{-x}\} \subset C(\mathbb{R})$, the subspace of the continuous functions on the real line spanned by the functions $1, e^x, e^{-x}$. Let $L(f(x)) = f'(x)$ and observe that $L(V) \subseteq V$. Find the matrix of $L : V \to V$ with respect to the basis v_1, v_2, v_3.

   Answer: The columns of the matrix of L are the coefficients of the images of the basis vectors. We compute $L(v_1) = 0$, $L(v_2) = v_2$, $L(v_3) = -v_3$, so the matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Let $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ be a vector not in the span of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Determine $\lim_{k\to\infty} A^k v$.

What is the limit for vectors in the span of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$?

Answer: This is a problem about eigenvalues and eigenvectors. First compute that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$, and a choice of associated eigenvectors is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. If we write $v = \alpha v_1 + \beta v_2$ then $A^k v = (1^k)\alpha v_1 + (\frac{1}{2})^k \beta v_2$, so $\lim_{k\to\infty} A^k v = \alpha v_1$. This shows that the limit is the zero vector for any vector in the span of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. To answer the first question, write

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a+2b}{3} v_1 + \frac{a-b}{3} v_2$$

so we conclude the limit for any other vector is $\frac{a+2b}{3} v_1$.

4. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$.

(a) What are the eigenvalues and a set of eigenvectors for A.

(b) Factor A into a product $A = XDX^{-1}$ where X is diagonal.

Answer:Compute the roots of $\det(A - \lambda Id)$ to obtain $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -2$, then compute the null spaces of $A - \lambda_j Id$ to get eigenvectors

$$v_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

thus $X = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 2 & 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, and we compute $X^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{12} & \frac{1}{4} & -\frac{3}{4} \end{pmatrix}$,

5. Let A be a 3×3 matrix such that $A^2 = Id$. What are the possible eigenvalues of A?

   Answer: If A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then A^2 has eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2$, but the eigenvalues of A^2 are all 1, we conclude $\lambda_j = \pm 1$.