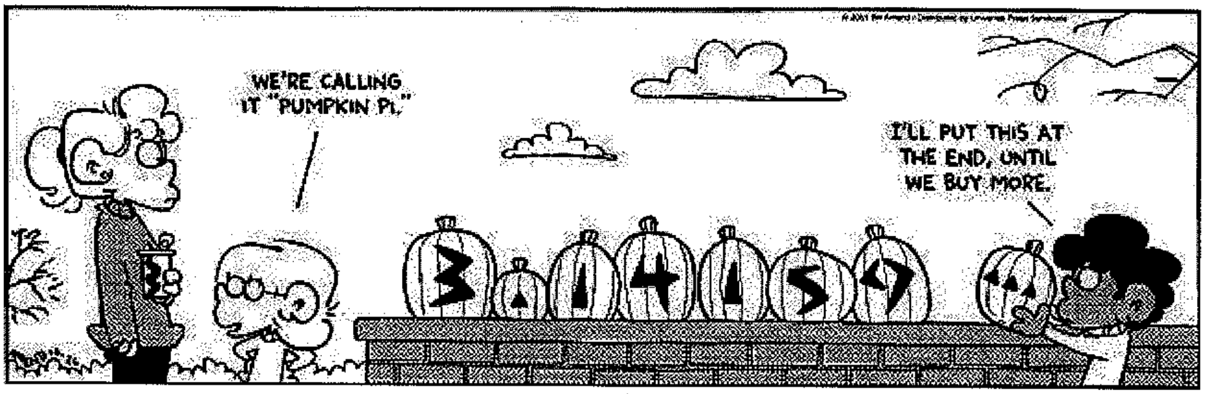
Math 308 Section 200 Exam 1B – Jean Marie Linhart Summer 2011



http://www.foxtrot.com

- You may use your 4x6 inch notecard for this exam. You must hand it in with your exam.
- You may not use any other notes, a calculator, or your book.
- You may not collaborate with your neighbors on this exam.
- You must show all appropriate work to receive credit, especially partial credit.
- If you use an integrating factor, identify it!
- The instructor will provide additional scratch paper if needed.
- Read each question carefully.

"An Aggie does not lie, cheat, or steal or tolerate those who do"
On my honor as an Aggie, I have neither given nor received unauthorized aid on this exam.

Printed name:______________________________

Signature:______________________________

Printed name: Jean Marie Linhart (1 point) (100 points + 1 extra)

1. (15 points) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 2 \end{bmatrix} \mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Identify clearly the general solution to 1st order ODE system and the solution to the IVP.

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 6 & 2-\lambda \end{bmatrix} = 0$$

$$(\lambda-1)(\lambda-2) - 6 = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda-4)(\lambda+1) = 0$$

$$\boxed{\lambda_1 = 4 \quad \lambda_2 = -1}$$

$\lambda_1 = 4$

$$\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + x_2 = 0$$

$$x_2 = 3x_1$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$\lambda_2 = -1$

$$\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$

$$x_2 = -2x_1$$

$$\boxed{\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{v}_2}$$

General Solution

$$\vec{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Specific Solution

$$x_1(t) = c_1 e^{4t} + c_2 e^{-t}$$

$$x_1(0) = 3 = c_1 + c_2$$

$$c_1 = 3 - c_2$$

$$x_2(t) = 3c_1 e^{4t} - 2c_2 e^{-t}$$

$$x_2(0) = -1 = 3c_1 - 2c_2$$

$$-1 = 3(3-c_2) - 2c_2$$

$$-1 = 9 - 5c_2$$

$$-10 = -5c_2$$

$$c_2 = 2 \quad c_1 = 1$$

$$\boxed{\begin{aligned} x_1 &= e^{4t} + 2e^{-t} \\ x_2 &= 3e^{4t} - 4e^{-t} \end{aligned}}$$

or

$$\boxed{\vec{x} = e^{4t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

2. (10 points) Use separation of variables to find an explicit solution to

$$\frac{dy}{dx} = 15x^4(y-3)^{2/3} \qquad y(0) = 3$$

$$\int (y-3)^{-2/3}\,dy = \int 15x^4\,dx$$

$$3(y-3)^{1/3} = 3x^5 + C$$

$$(y-3)^{1/3} = x^5 + C$$

$$y - 3 = (x^5+C)^3$$

$$y(x) = (x^5+C)^3 + 3$$

$$y(0) = 3 = C^3 + 3$$

$$C = 0$$

$$\boxed{y(x) = x^{15} + 3}$$

(7 points) Is the solution you found by separation of variables a unique solution? If so, give a theorem stating why. If not, give another solution and an explanation of why the theorem about uniqueness doesn't apply.

For uniqueness $\frac{dy}{dx} = F(x,y)$ must have both $F(x,y)$ continuous and $\frac{\partial}{\partial y}F(x,y)$ continuous. In this problem $F(x,y) = 15x^4(y-3)^{2/3}$ is continuous, but $\frac{\partial F}{\partial y} = \frac{2}{3}(15x^4)(y-3)^{-1/3}$ is <u>not</u> continuous at $y=3$.

For $y=3$ $F(x,y)=0$ and $\frac{dy}{dx} = 0 \Rightarrow$

$\boxed{y(x) = 3}$ is another solution

constant solution

3. (2 points) Classify this equation. (homogeneous? autonomous? linear? separable? what order?)

$$-y = x^3\cos(x) - xy' \qquad y(\pi) = 0$$

$xy' - y = x^3 \cos x$

$y' - \frac{1}{x} y = x^2 \cos x$

Not homogeneous
Not autonomous
1st order linear
Not separable

(15 points) Find an explicit solution to the initial value problem given above.

$\mu(x) = \exp \int -\frac{1}{x} dx = \exp(\ln x^{-1}) = x^{-1}$

$x^{-1} y = \int x^{-1}(x^2 \cos x)\, dx + C$

$= \int x \cos x \, dx + C = x \sin x - \int \sin x \, dx + C$

$u = x \qquad dv = \cos x \, dx$

$du = dx \qquad v = \sin x$

$x^{-1} y = x \sin x + \cos x + C$

$y = x^2 \sin x + x \cos x + Cx$

$y(\pi) = 0 = -\pi + C\pi \Rightarrow C = 1$

$$\boxed{y(x) = x^2 \sin x + x \cos x + x}$$

4. Given

$$(3y^3 \sin x \cos x - 8y^2 \cos x) + (3y^2 \sin^2 x - 8y \sin x)\frac{dy}{dx} = 0$$

$$\underbrace{(3y^3 \sin x \cos x - 8y^2 \cos x)}_{M} \qquad \underbrace{(3y^2 \sin^2 x - 8y \sin x)}_{N}$$

(8 points) Show this equation is not exact, and show how to find the integrating factor. The integrating factor is $\sin x$.

$$\frac{\partial M}{\partial y} = 9y^2 \sin x \cos x - 16y \cos x \neq \frac{\partial N}{\partial x} = 6y^2 \sin x \cos x - 8y \cos x$$

Not exact!

$$\frac{d\mu(x)}{\mu} = \frac{M_y - N_x}{N} = \frac{9y^2 \sin x \cos x - 16y\cos x - 6y^2 \sin x \cos x + 8y\cos x}{y \sin x\,(3y \sin x - 8)}$$

$$\frac{d\mu}{\mu} = \frac{3y^2 \sin x \cos x - 8y\cos x}{y \sin x (3y \sin x - 8)} = \frac{\cancel{y}\cos x\,\cancel{(3y \sin x - 8)}}{\cancel{y}\sin x\,\cancel{(3y \sin x - 8)}} = \frac{\cos x}{\sin x}$$

$$\int \frac{d\mu}{\mu} = \int \frac{\cos x}{\sin x}\,dx \Rightarrow \ln \mu = \ln \sin x$$

$u = \sin x$

$du = \cos x$

$$\boxed{\mu(x) = \sin x}$$

(8 points) Use the integrating factor from above to make the equation exact; show that it is **now** exact, and find an implicit solution.

$$\underbrace{3y^3 \sin^2 x \cos x - 8y^2 \sin x \cos x}_{M} + (\underbrace{3y^2 \sin^3 x - 8y \sin^2 x}_{N})\frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = 9y^2 \sin^2 x \cos x - 16y \sin x \cos x \qquad \frac{\partial N}{\partial x} = 9y^2 \sin^2 x \cos x - 16y \sin x \cos x$$

Exact!

$$\int 3y^3 \sin^2 x \cos x - 8y^2 \sin x \cos x\,dx = y^3 \sin^3 x - 4y^2 \sin^2 x + g(y)$$

$u = \sin x \quad du = \cos x\,dx$

$$\int 3y^2 \sin^3 x - 8y \sin^2 x\,dy = y^3 \sin^3 x - 4y^2 \sin^2 x + f(x)$$

$\Rightarrow$ Solution is $\boxed{y^3 \sin^3 x - 4y^2 \sin^2 x = C}$

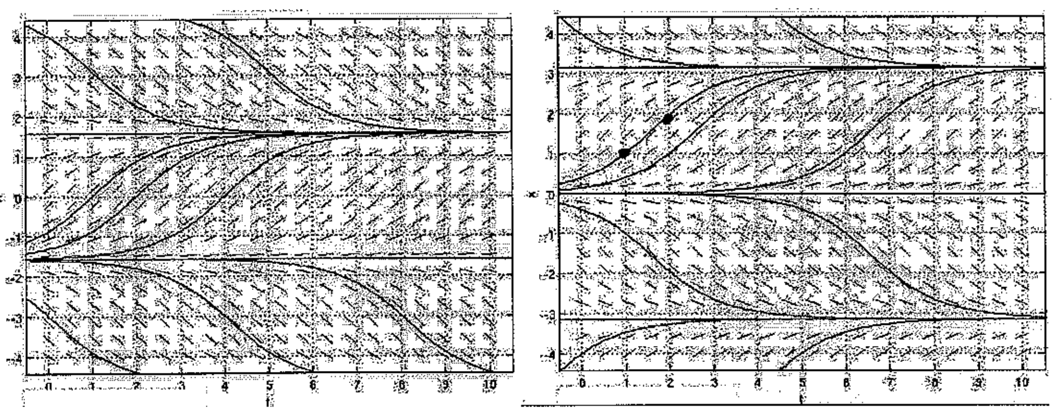
5. The following two direction fields are for the differential equations

$$\text{Equation A:} \quad \frac{dx}{dt} = \cos(x)$$

$$\text{Equation B:} \quad \frac{dx}{dt} = \sin(x)$$

(4 points) Identify which direction field goes with which equation.

Equation A



Equation B

(2 points) Use the correct direction field and solution passing through $x(1) = 1$ to approximate $x(2)$ for **equation B.**

$x(2) \approx$ 1.8 or 1.9

(4 points) Identify the equilibria for **equation B** and which equilibria are stable and which are unstable.

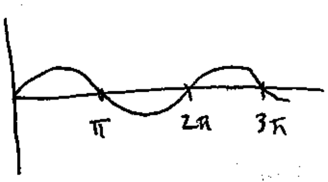
$\frac{dx}{dt} = \sin x$

$\sin x = 0$ at $\ldots, -\pi, 0, \pi, 2\pi, \ldots$

So equilibria are $\{n\pi\}$ $n \in \mathbb{Z}$

Stable equilibria at $\{(2n+1)\pi\}$ $n \in \mathbb{Z}$

(odd values of π



6. A spherical hailstone melts with the change in its volume proportional to its surface area. Assume it remains spherical as it melts. Initially it is $\frac{4}{3}\pi$ cubic inch in **volume**, and after 6 minutes it is $\frac{4}{81}\pi$ cubic inch in **volume**.

(6 points) Set up a differential equation with initial condition representing the change in the **radius** of the hailstone as a function of time.

The volume of a sphere is $V = \frac{4}{3}\pi r^3$, and its surface area is $S = 4\pi r^2$.

$$\frac{dV}{dt} = KS \qquad V(0) = \frac{4\pi}{3}(\text{in})^3 = \frac{4}{3}\pi [r(0)]^3 \Rightarrow r(0) = 1 \text{ in}$$

$$\frac{d\left(\frac{4}{3}\pi r^2\right)}{dt} = 4\pi r^2 \frac{dr}{dt} = K(4\pi r^2)$$

$$\boxed{\frac{dr}{dt} = K \qquad r(0) = 1}$$

(6 points) Solve the differential equation from above and use the information given to write an explicit equation for the **radius** as a function of time.

$$\int dr = \int K\,dt$$

$$r = Kt + C$$

$$r(0) = 1 \Rightarrow C = 1$$

$$r(6) = 1/3 = K(6) + 1$$

$$-2/3 = 6K$$

$$K = -1/9$$

$$V(6) = \frac{4}{81}\pi (\text{in})^3 = \frac{4}{3}\pi [r(6)]^3 \text{ in}^3$$

$$r(6) = \left[\frac{3}{4} \cdot \frac{4}{81}\right]^{1/3} \text{in}$$

$$r(6) = \left[\frac{1}{27}\right]^{1/3} \text{in} = 1/3 \text{ in}$$

$$\boxed{r(t) = -1/9\, t + 1}$$

(5 points) Use your solution, above, to calculate when the hailstone completely disappears. Do not forget to include the correct **unit** in your answer.

Find t where $r = 0$

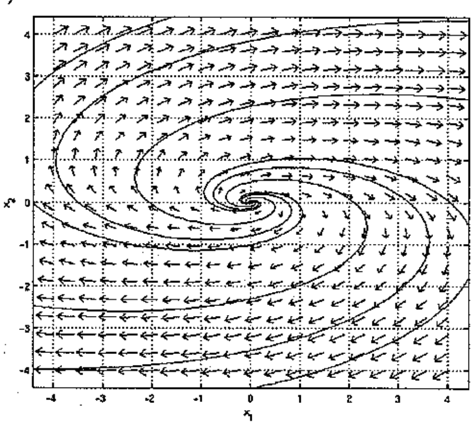
$$0 = -1/9\, t + 1$$

$$\boxed{t = 9 \text{ minutes}}$$

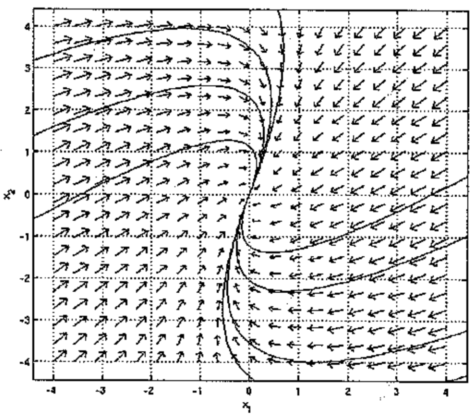
7. (8 points) Each eigenvalue/eigenvector pair comes from a 2-dimensional linear system of the form $\mathbf{x}' = A\mathbf{x}$. Give the letter for the correct phase plane for each eigenvalue/eigenvector pair. If there is no correct phase plane, choose "None of these phase planes". Answers may be used more than once or not at all!

i) $\lambda_1 = -4 \quad \mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ $\lambda_2 = -2 \quad \mathbf{v}_2 = \begin{bmatrix}1\\3\end{bmatrix}$ (B)	ii) $\lambda_1 = 1-2i \quad \mathbf{v}_1 = \begin{bmatrix}1\\-0.5i\end{bmatrix}$ $\lambda_2 = 1+2i \quad \mathbf{v}_2 = \begin{bmatrix}1\\+0.5i\end{bmatrix}$ (A)
iii) $\lambda_1 = i \quad \mathbf{v}_1 = \begin{bmatrix}1\\0.4-0.2i\end{bmatrix}$ $\lambda_2 = -i \quad \mathbf{v}_2 = \begin{bmatrix}1\\0.4+0.2i\end{bmatrix}$ (E)	iv) $\lambda_1 = 0 \quad \mathbf{v}_1 = \begin{bmatrix}4\\1\end{bmatrix}$ $\lambda_2 = 3 \quad \mathbf{v}_2 = \begin{bmatrix}-2\\1\end{bmatrix}$ (E)

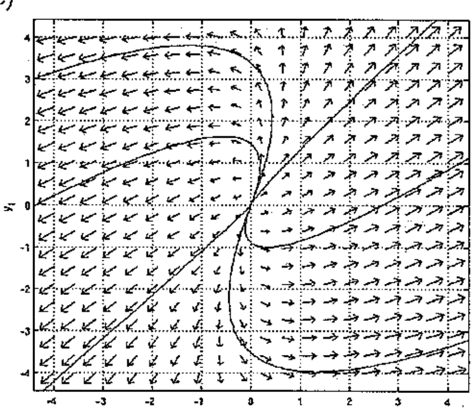
a)



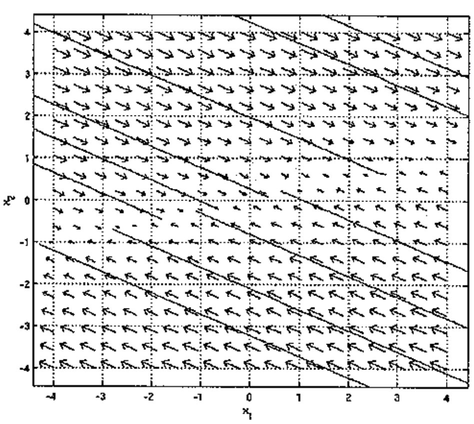
b)



c)



d)



e)
None of these phase planes.