**#1 blue test and #3 yellow test**
(20 points) Find an implicit solution to the following:

$$\frac{dy}{dx} = -\frac{y^3 + 4e^x y}{2e^x + 3y^2}$$

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**Solution:** Rewrite this in the form:

$$M(x,y)dx + N(x,y)dy = 0$$

we get

$$(y^3 + 4e^x y)dx + (2e^x + 3y^2)dy = 0$$

For exactness check to see if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is

$$3y^2 + 4e^x \neq 2e^x$$

this is not exact.
Find a special integrating factor to make this exact:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y^2 + 2e^x}{2e^x + 3y^2} = 1$$

$$\mu(x) = \exp\left[\int 1 dx\right] = e^x$$

$e^x$ is a safe special integrating factor; no solutions are gained or lost.
The new equation is

$$(y^3 e^x + 4e^{2x} y)dx + (2e^{2x} + 3e^x y^2)dy = 0$$

This is $M_1 dx + N_1 dy = 0$ and now check for exactness with $M_1$ and $N_1$. It is exact:

$$\frac{\partial M_1}{\partial y} = 3y^2 e^x + 4e^{2x} = 4e^{2x} + 3y^2 e^x = \frac{\partial N_1}{\partial x}$$

Integrate

$$F(x,y) = \int M_1(x,y)dx = \int (y^3 e^x + 4e^{2x} y)dx = y^3 e^x + 2e^{2x} y + g(y)$$

Equate

$$\frac{\partial F}{\partial y} = N_1(x,y)$$

This gives

$$3y^2 e^x + 2e^{2x} + g'(y) = 2e^{2x} + 3e^x y^2 \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = C$$

Incorporate this constant with the constant in the implicit solution, giving:

$$F(x,y) = y^3 e^x + 2e^{2x} y = C$$

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**#2 both tests**
(15 points) Find an explicit solution to the following:

$$x\frac{dy}{dx} - y = x^3 \sin(2x) \qquad y(\pi) = 0$$

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**Solution:** Write this in standard form by dividing out the leading $x$:

$$\frac{dy}{dx} - \frac{y}{x} = x^2 \sin(x)$$

This is linear. Find the integrating factor

$$\mu(x) = \exp\left[\int -\frac{1}{x} dx\right] = \exp(-\ln(x)) = x^{-1}$$

Multiply through by the integrating factor and integrate

$$yx^{-1} = \int x \sin(2x) dx$$

Integrate by parts with $u = x$, $du = dx$, and $dv = \sin(2x)dx$, $v = -(1/2)\cos(2x)$ to obtain

$$yx^{-1} = -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$$

Solve for $y$

$$y = -\frac{1}{2}x^2\cos(2x) + \frac{x}{4}\sin(2x) + Cx$$

Now use the initial condition

$$y(\pi) = 0 = -\frac{1}{2}\pi^2 + C\pi$$

to solve for $C$:

$$C = \frac{1}{2}\pi$$

Write down the final answer

$$y(x) = -\frac{1}{2}x^2\cos(2x) + \frac{x}{4}\sin(2x) + \frac{1}{2}\pi x$$

**#3 blue test and #4 yellow test**
(2 points) Identify the dependent and independent variables in:

$$\frac{dy}{d\theta} + \frac{y}{\theta} = -6\theta y^{-2} \qquad y(1) = 0$$

Independent variable: $\theta$ Dependent variable: $y$

(2 points) What kind of a differential equation is this?

This is a Bernoulli equation

(15 points) Find an explicit solution to the above initial value problem.

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**Solution:** Rewrite:

$$y^2 \frac{dy}{d\theta} + \frac{y^3}{\theta} = -6\theta$$

Now we try substituting $y^3 = v$. Differentiate this with respect to $\theta$

$$\frac{dv}{d\theta} = 3y^2 \frac{dy}{d\theta}$$

This gives a new equation of

$$\frac{1}{3}\frac{dv}{d\theta} + \frac{v}{\theta} = -6\theta$$

Do not forget to get rid of the 1/3 out front!

$$\frac{dv}{d\theta} + \frac{3v}{\theta} = -18\theta$$

Now, find the integrating factor

$$\mu(\theta) = \exp\left[\int \frac{3}{\theta} d\theta\right] = \exp\left[3\ln(\theta)\right] = \exp\left[\ln(\theta^3)\right] = \theta^3$$

This gives

$$\theta^3 v = \int -18\theta^4 d\theta = -\frac{18}{5}\theta^5 + C \quad \Rightarrow \quad v = -\frac{18}{5}\theta^2 + \frac{C}{\theta^3}$$

Substitute $v = y^3$ and solve for $y$:

$$y^3 = -\frac{18}{5}\theta^2 + \frac{C}{\theta^3} \quad \Rightarrow \quad y = \left[-\frac{18}{5}\theta^2 + \frac{C}{\theta^3}\right]^{1/3}$$

Use the initial condition

$$y(1) = 0 = \left[-\frac{18}{5} + C\right]^{1/3} \quad \Rightarrow \quad C = \frac{18}{5}$$

Substituting this into our equation for $y$ we obtain

$$y(\theta) = \left[-\frac{18}{5} + \frac{18}{5\theta^3}\right]^{1/3}$$

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**#4 blue test and #5 yellow test**
(10 points) Use separation of variables to find the solution to

$$\frac{dy}{dx} = 15x^4(y-1)^{2/3} \qquad y(0) = 1$$

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**Solution:**

$$\int \frac{dy}{(y-1)^{2/3}} = \int 15x^4 dx$$

$$3(y-1)^{1/3} = 3x^5 + C$$

Divide through by 3 and rename the constant

$$(y-1)^{1/3} = x^5 + C$$

Take the cube of both sides

$$y - 1 = (x^5 + C)^3 \quad \Rightarrow \quad y = (x^5 + C)^3 + 1$$

$$y(0) = 1 \quad \Rightarrow \quad C^3 + 1 = 1 \quad \Rightarrow \quad C = 0$$

$$y = x^{15} + 1$$

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(5 points) Is the solution you found by separation of variables a unique solution? If so, give a theorem stating why. If not, give another solution.

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**Solution:** No, this solution is not unique. We have

$$\frac{dy}{dx} = f(x,y) = 15x^4(y-1)^{2/3}$$

which is continuous everywhere, and

$$\frac{\partial f}{\partial y}(x,y) = 10x^4(y-1)^{-1/3}$$

which is not continuous at $y = 1$. This suggests we look at

$$y(x) \equiv 1$$

for our second solution. It is another solution.

**#5 blue test and #6 yellow test**

(5 points) An ice cube (all sides are square) melts with the change in its volume proportional to its surface area. Assume it remains cubical as it melts.

**Blue test:** Initially it is 1 cubic inch in **volume**, and after 5 minutes it is 1/8 cubic inch in **volume**.

**Yellow test:** Initially it is 1 cubic inch in **volume**, and after 5 minutes it is 1/27 cubic inch in **volume**.

Set up a differential equation representing the change in the length of a side of the ice cube as a function of time. Note: a cube has 6 faces that are all perfect squares of the same size.

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**Solution:** The differential equation in terms of $V$ for volume and $S$ for surface area is

$$\frac{dV}{dt} = kS$$

Substituting $V = s^3$ and $S = 6s^2$ where $s$ is the length of a side of the cube, rewrite this in terms of $s(t)$:

$$3s^2\frac{ds}{dt} = 6ks^2 \quad \Rightarrow \quad \frac{ds}{dt} = 2k \qquad \text{or} \qquad \frac{ds}{dt} = k$$

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(5 points) Solve the differential equation from above and use the information given to write an explicit equation for the **length of a side of the cube** as a function of time.

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**Solution:** Multiply through by $dt$

$$ds = kdt \qquad \text{integrate both sides} \qquad s = kt + C$$

Use the conditions given in the problem. Initially $V(0) = 1 = [s(0)]^3$

$$s(0) = 1 \quad \Rightarrow \quad C = 1$$

**Blue test** At 5 minutes the volume is $V = 1/8$ or $s = 1/2$ use this to get $k$

$$s(5) = 1/2 = 5k + 1 \quad \Rightarrow \quad 5k = -\frac{1}{2} \quad \Rightarrow \quad k = -\frac{1}{10} \quad \Rightarrow \quad s(t) = -\frac{1}{10}t + 1$$

**Yellow test:** At 5 minutes the volume is $V = 1/27$ or $s = 1/3$ use this to get $k$

$$s(5) = 1/3 = 5k + 1 \quad \Rightarrow \quad 5k = -\frac{2}{3} \quad \Rightarrow \quad k = -\frac{2}{15} \quad \Rightarrow \quad s(t) = -\frac{2}{15}t + 1$$

(5 points) Use your solution, above, to calculate when the ice cube completely disappears.

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**Solution:** The ice cube completely disappears when the length of a side is 0, which means

$$s(t) = 0$$

we want to calculate $t$

**Blue test:**

$$s(t) = 0 = -\frac{1}{10}t + 1 \Rightarrow -\frac{1}{10}t = -1 \Rightarrow t = 10\text{minutes}$$

**Yellow test:**

$$s(t) = 0 = -\frac{2}{15}t + 1 \Rightarrow -\frac{2}{15}t = -1 \Rightarrow t = \frac{15}{2}\text{minutes}$$

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**#6 blue test #7 yellow test** (12 points) Find any explicit solution for

$$\frac{dy}{dx} = 2xy^2 + 2x \qquad y(\sqrt{\pi}) = 0$$

This problem is unusual in that there are many different equivalent solutions, i.e. there are many constants that will work. You merely need to find one constant that works to make this equation true.

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**Solution:** Separate the variables

$$\int \frac{dy}{y^2+1} = \int 2x dx$$

Integrate

$$\arctan(y) = x^2 + C \quad \Rightarrow \quad y = \tan(x^2 + C)$$

The initial condition is

$$y(\sqrt{\pi}) = 0 = \tan(\pi + C)$$

This is solved by any integer multiple of $\pi$, i.e. $C = k\pi$. In particular $C = 0$ works just fine and

$$y(x) = \tan(x^2)$$

is a solution.

**#7 blue test #1 yellow test**

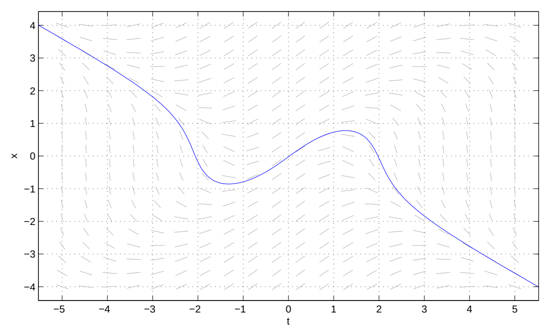
The following two direction fields are for the differential equations

$$\text{Equation A:} \quad \frac{dx}{dt} = \frac{t^2}{1+x^2}$$

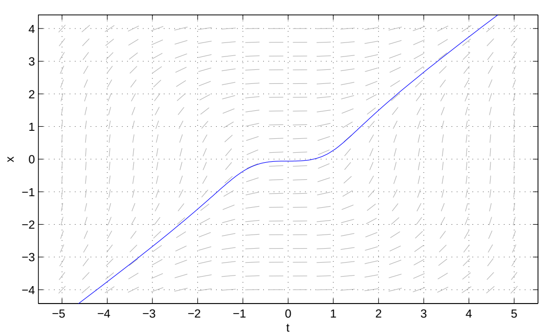
$$\text{Equation B:} \quad \frac{dx}{dt} = \frac{-t^2}{1+x^2} + 1$$

(2 points) Identify which direction field goes with which equation.

**Solution: this is equation B**




**Solution: this is equation A**




**Blue test** (2 points) Use the correct direction field and solution passing through $x(0) = 0$ to approximate $x(2)$ for **equation A**.

$x(2) \approx$ ______1.5______

**Yellow test** (2 points) Use the correct direction field and solution passing through $x(0) = 0$ to approximate $x(2)$ for **equation B**.

$x(2) \approx$ ______0______

**#8 both tests**
(5 points extra credit) Let $y' = F(x, y)$ be an ordinary differential equation where $F(x, y)$ is continuous and differentiable with respect to $x$ and $y$ **with continuous derivatives with respect to** $x$ **and** $y$ for all $x$ and $y$. Let $y_1(x)$ and $y_2(x)$ be two solutions to this ODE. If $y_1(0) > y_2(0)$ Is it possible that $y_1(1) < y_2(1)$? Why or why not?

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**Solution:** Note this problem has changed slightly see **bold text** above. Without assuming continuous derivatives on $F(x, y)$, it was harder to make an argument here.

Let us assume there exists $y_1(x)$ and $y_2(x)$ that solve $y' = F(x, y)$ with $y_1(0) > y_2(0)$ and $y_1(1) < y_2(1)$. Since $F(x, y)$ is continuous, these solutions must be continuous, and this implies that there exists $x_0 \in (0, 1)$ with $y_1(x_0) = y_2(x_0)$, i.e. this implies the two must cross if they change positions. Let's call $y_1(x_0) = y_2(x_0) = y_0$.

Now, on the other hand,if we have $y_1(x_0) = y_2(x_0) = y_0$ and $F(x, y)$ is differentiable for all $x, y$ with a continuous derivative, any solution to the initial value problem $y' = F(x, y)$ with $y(x_0) = y_0$ is unique, by the uniqueness theorem. This then implies that $y_1(x) = y_2(x)$ for all values of $x$. This is a contradiction.

Therefore our assumption was incorrect. We cannot have $y_1(1) < y_2(1)$ if $y_1(0) > y_2(0)$.