#1 yellow test and #2 blue test
(10 points) Find an explicit solution to the following:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0 \qquad y(0) = 0, \quad y'(0) = -2\sqrt{2}$$

Hint: Use the quadratic formula.

Solution: The auxiliary equation is $r^2 - 2r + 3 = 0$. Note that this **does not factor** into $(r-3)(r+1) = 0$! Using the quadratic formula we find the roots

$$r = \frac{+2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm i\sqrt{2}$$

The fundamental solutions are

$$e^x \cos(x\sqrt{2}) \qquad \text{and} \qquad e^x \sin(x\sqrt{2})$$

with general form of the solution

$$y(x) = c_1 e^x \cos(x\sqrt{2}) + c_2 e^x \sin(x\sqrt{2})$$

We use the initial condition to solve for c_1 and c_2. First we calculate $y'(x)$.

$$y'(x) = c_1 e^x \cos(x\sqrt{2}) - c_1\sqrt{2}e^x \sin(x\sqrt{2}) + c_2 e^x \sin(x\sqrt{2}) + c_2\sqrt{2}e^x \cos(x\sqrt{2})$$

now

$$y(0) = 0 = c_1$$
$$y'(0) = c_2\sqrt{2} = -2\sqrt{2} \Rightarrow c_2 = -2$$

Our solution is

$$y(x) = -2e^x \sin(x\sqrt{2})$$

#2 yellow test and #1 blue test
(10 points) Find the general solution of the Cauchy-Euler equation

$$x^2y'' - 3xy' + 4y = 0$$

Solution: We use the transformation $x = e^t$. Using the chain rule, we calculate the derivatives of y with respect to t:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = x\frac{dy}{dx}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

or

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Plugging this into our original equation, it becomes:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 4y = 0$$

which is

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$$

Our auxiliary equation is

$$r^2 - 4r + 4 = (r-2)^2 = 0$$

with repeated root $r = 2$. Fundamental solutions are e^{2t} and te^{2t}

$$y(t) = c_1e^{2t} + c_2te^{2t}$$

Switching back to x, we have

$$y(x) = c_1x^2 + c_2x^2\ln(x)$$

(10 points) Now use your work from above and find the general solution for

$$x^2y'' - 3xy' + 4y = x^2$$

Solution: Switching this equation entirely over to the t variable we have

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^{2t}$$

I will solve this easily by the method of undetermined coefficients. The guess for $L[y] = e^{2t}$ is At^se^{2t}. I must choose $s = 2$ since e^{2t} amd te^{2t} were solutions to the homogeneous equation. The particular solution is $y_p(t) = At^2e^{2t}$. Using the product rule I find its derivatives

$$y_p'(t) = 2Ate^{2t} + 2At^2e^{2t} \qquad \text{and} \qquad y_p'' = 2Ae^{2t} + 8Ate^{2t} + 4At^2e^{2t}$$

Plugging in to $L[y]$:

$$2Ae^{2t} + 8Ate^{2t} + 4At^2e^{2t} - 4\left[2Ate^{2t} + 2At^2e^{2t}\right] + 4At^2e^{2t} = e^{2t}$$

Note that all the terms in t^2 and t cancel, leaving me with

$$2Ae^{2t} = e^{2t} \quad \Rightarrow \quad A = \frac{1}{2}$$

My general solution is

$$y_g(t) = \frac{1}{2}t^2e^{2t} + c_1e^{2t} + c_2te^{2t}$$

Switching back to the x variable, I get

$$y_g(x) = \frac{1}{2}x^2[\ln(x)]^2 + c_1x^2 + c_2x^2[\ln(x)]$$

#3 yellow test and #4 blue test
(8 points) According to the existence and uniqueness theorem for 2nd order linear ODEs, on what intervals might the following ODE have unique solutions?

$$(x^2 - 3x)y'' + 2xy' - y = x^2$$

Solution: We need this in the form

$$y'' + p(x)y' + q(x)y = g(x)$$

Divide through by $x^2 - 3x = x(x-3)$ in front of y'' to get

$$y'' + \frac{2}{x-3}y' - \frac{y}{x(x-3)} = \frac{x}{x-3}$$

The denominators have problems at $x = 0$ and/or $x = 3$.

Therefore unique solutions exist on the largest intervals not containing these points, which are

$$(-\infty, 0), \quad (0, 3), \quad \text{and} \quad (3, \infty)$$

#4 yellow test and #5 blue test
(5 points) **YOU DO NOT HAVE TO SOLVE THIS.** If you were using the method of undetermined coefficients to solve this equation, what form of a particular solution would you attempt (guess) and **why**?

$$y'' + 2y' + y = e^{-x}$$

Solution: My guess is Ax^se^{-x}, but I have to determine s.

The auxiliary equation is $r^2 + 2r + 1 = (r+1)^2 = 0$ with repeated root equal to $r = -1$. Fundamental solutions to the homogeneous equation are e^{-x} and xe^{-x}.

Therefore, I can't pick $s = 0$ or $s = 1$ or Ax^se^{-x} would be a solution to the homogeneous equation.

Ax^2e^{-x} would be my guess.

#5 yellow test and #6 blue test
(3 points) Use one step of the Euler method to calculate an approximation to $y(1.1)$ for the IVP

$$y' = 2x^2 - xy, \qquad y(1) = 2$$

Solution: $x_0 = 1$, $y_0 = 2$ and $f(x, y) = 2x^2 - xy$.

$$y_1 = y_0 + h[f(x_0, y_0)] = 2 + 0.1[2 - 2] = 2$$

(4 points) Use one step of the Improved Euler Method to calculate an approximation to $y(1.1)$ for the IVP above.

Solution: y_1^p in the formula below is calculated from the Euler method, so it is the solution of the previous problem.

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^p)] = 2 + 0.05[0 + 2(1.1)^2 - (1.1)(2)]$$

$$y_1 = 2 + 0.05[2.42 - 2.2] = 2.011$$

#6 yellow test and #3 blue test
(18 points) Find the solution to the IVP

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = \frac{34}{3}\cos(x) \qquad y(0) = 1, \quad y'(0) = 1$$

Solution: The auxiliary equation is $r^2 - 5r + 4 = 0$ which factors (nicely!) into $(r - 4)(r - 1) = 0$, and has solutions $r = 1, 4$. The fundamental solutions to the homogeneous equation are $y_1 = e^{4x}$ and $y_2 = e^x$.
Using the method of undetermined coefficients, my guess is

$$y_p = A\cos(x) + B\sin(x)$$

$$y_p' = -A\sin(x) + B\cos(x)$$
$$y_p'' = -A\cos(x) - B\sin(x)$$

Plugging this back into the differential equation, I get

$$-A\cos(x) - B\sin(x) - 5[-A\sin(x) + B\cos(x)] + 4[A\cos(x) + B\sin(x)] = \frac{34}{3}\cos(x)$$

I collect terms in $\cos(x)$ and get the first equation; I collect terms in $\sin(x)$ and get the second equation.

$$\begin{cases} 3A - 5B &= \frac{34}{3} \\ 5A + 3B &= 0 \end{cases}$$

Solving this system of equations yields $A = 1$ and $B = -5/3$.
The general solution is

$$y(x) = \cos(x) - \frac{5}{3}\sin(x) + c_1 e^{4x} + c_2 e^x$$

with derivative

$$y'(x) = -\sin(x) - \frac{5}{3}\cos(x) + 4c_1 e^{4x} + c_2 e^x$$

The initial conditions give

$$y(0) = 1 = 1 + c_1 + c_2 \quad \Rightarrow \quad c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$

and

$$y'(0) = 1 = -\frac{5}{3} + 4c_1 + c_2 \quad \Rightarrow \quad \frac{8}{3} = 4c_1 + c_2$$

$$\frac{8}{3} = 3c_1 \quad \Rightarrow \quad c_1 = \frac{8}{9} \quad \text{and} \quad c_2 = -\frac{8}{9}$$

$$y(x) = \cos(x) - \frac{5}{3}\sin(x) + \frac{8}{9}e^{4x} - \frac{8}{9}e^x$$

#7 yellow test and #8 blue test
(15 points) Find the general solution to

$$y'' + 4y' + 4y = e^{-2x}\ln(x)$$

Solution: The auxiliary equation is $r^2+4r+4 = (r+2)^2 = 0$ solved by $r = -2$. Solutions to the homogeneous equation are $y_1(x) = e^{-2x}$ and $y_2 = xe^{-2x}$.

I will solve this (I have to) by variation of parameters. You **cannot** use undetermined coefficients on this.

I try to find a particular solution of the form $v_1y_1 + v_2y_2$, where v_1 and v_2 are unknown functions of x.

The formulas for v_1 and v_2 are

$$v_1(x) = \int \frac{-y_2 g}{y_1y_2' - y_2y_1'}dx \quad \text{and} \quad v_2(x) = \int \frac{y_1 g}{y_1y_2' - y_2y_1'}dx$$

I will need $y_1y_2' - y_2y_1'$. $y_1' = -2e^{-2x}$ and $y_2' = e^{-2x} - 2xe^{-2x}$. I get

$$y_1y_2' - y_2y_1' = e^{-4x}$$

Substituting into my formulas

$$v_1 = \int -x\ln(x)dx = -\frac{x^2}{2}\ln(x) + \int \frac{x}{2}dx = \frac{x^2}{4} - \frac{x^2}{2}\ln(x)$$

To solve this I used integration by parts with $u = \ln(x)$, $du = (1/x)dx$, $dv = -xdx$ and $v = -(x^2/2)$.

$$v_2 = \int \ln(x)dx = x\ln(x) - x$$

To solve this I used the same integration by parts technique with $u = \ln(x)$. The particular solution $v_1y_1 + v_2y_2$ is

$$y_p(x) = \frac{x^2}{2}\left[\frac{1}{2} - \ln(x)\right]e^{-2x} + x^2\left[\ln(x) - 1\right]e^{-2x} = \frac{x^2}{2}e^{-2x}\left[\ln(x) - \frac{3}{2}\right]$$

And the general solution is

$$y_g(x) = \frac{x^2}{2}e^{-2x}\left[\ln(x) - \frac{3}{2}\right] + c_1e^{-2x} + c_2xe^{-2x}$$

#8 yellow test and #7 blue test
(5 points) Show the Laplace transform of $f(t) = \sin(bt)$ is

$$\mathcal{L}\{f\}(s) = \frac{b}{s^2 + b^2}$$

by using the integral definition of the Laplace transform and performing the integral. You may use the integration formula

$$\int e^{cx}\sin(ax)dx = \frac{1}{a^2 + c^2}e^{cx}\left[c\sin(ax) - a\cos(ax)\right] + C$$

rather than integrating by parts twice!

Solution:

$$\begin{aligned}\int_0^\infty e^{-st}\sin(bt)dt &= \frac{1}{b^2 + s^2}e^{-st}\left[-s\sin(bt) - b\cos(bt)\right]\Big|_0^\infty \\ &= \frac{-1}{b^2 + s^2}(-b) = \frac{b}{b^2 + s^2}\end{aligned}$$

(2 points) For what values of s does $\mathcal{L}\{f\}(s)$ exist for $f(t) = \sin(bt)$?

Solution: $s > 0$

(5 points) Use the integral definition of Laplace transform to show that if $\mathcal{L}\{g(t)\}(s) = G(s)$ then

$$\mathcal{L}\{e^{at}g(t)\}(s) = G(s-a)$$

Solution:

$$\mathcal{L}\{g(t)\}(s) = G(s) = \int_0^\infty e^{-st}g(t)dt$$

$$\mathcal{L}\{e^{at}g(t)\}(s) = \int_0^\infty e^{-st}e^{at}g(t)dt = \int_0^\infty e^{-(s-a)t}g(t)dt = G(s-a)$$

Wherever we saw an s in the first equation, we had an $s-a$ in the second.

(5 points) What is

$$\mathcal{L}\{7e^{3t}\sin(2t) - 3e^{4t}\}?$$

You do not need to use the integral definition of the Laplace transform, but you must show your work!

Solution:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

and, from above

$$\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$$

Now, using these and the linearity of the Laplace transform, I get

$$\mathcal{L}\{7e^{3t}\sin(2t) - 3e^{4t}\} = 7\mathcal{L}\{e^{3t}\sin(2t)\} - 3\mathcal{L}\{e^{4t}\} = \frac{14}{(s-3)^2+4} - \frac{3}{s-4}$$

#9 both tests

Extra credit. Choose and do **at most one** of the following.

a. (5 points extra credit) The variation of parameters method assumes you have $y_1(x)$ and $y_2(x)$ which are linearly independent solutions to the homogeneous equation

$$L[y] = y'' + p(x)y' + q(x)y = 0$$

then in order to solve

$$L[y] = y'' + p(x)y' + q(x)y = g(x)$$

you attempt a solution of the form $v_1(x)y_1(x) + v_2(x)y_2(x) = y_p(x)$ with the restriction that $v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0$. Using this and the differential equation above, derive the two formulas for $v_1(x)$ and $v_2(x)$.

or

b. (3 points extra credit) Return to your solution to problem 1, which was to find a solution to the IVP

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0 \qquad y(0) = 0, \quad y'(0) = -2\sqrt{2}$$

Would it make sense for this equation to represent a mass-spring oscillator in the real world? Why or why not?

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a. This was done in class and is done in your book.

b. This is not a mass spring oscillator in the real world. You could see this either of two ways.
(1) As $x \to \infty$ the solution gets larger oscillations, not smaller.
(2) The coefficient for damping by friction or air resistance has the wrong sign. Consequently, instead of damping, it is increasing the osciallations.