Printed name: Jean Marie (1 point) (100 points + 1 extra)

1. (9 points) Transform this 2nd order linear differential equation

$$t^2y'' + 7ty' + 9y = t^{-3} \qquad y(1) = 1, \quad y'(1) = 2$$

into a first order system of equations of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t) \qquad \mathbf{x}(1) = \mathbf{x}_0$$

Explicitly identify the matrix $A(t)$ and the vectors $\mathbf{g}(t)$ and $\mathbf{x}_0$.

yellow exam #6

$$t^2y'' = -9y - 7ty' + t^{-3} \qquad x_1 = y, \; x_2 = y'$$

$$y'' = -\frac{9}{t^2}y - \frac{7}{t}y' + t^{-5} \qquad x_2' = -9/t^2\, x_1 - \frac{7}{t}x_2 + t^{-5}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -9/t^2 & -7/t \end{bmatrix}}_{A(t)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ t^{-5} \end{bmatrix}}_{\vec{g}(t)}$$

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{x}_0}$$

2. (8 points) According to the existence and uniqueness theorem for 2nd order linear ODEs, on what intervals might the following ODE have unique solutions?

$$(t^2 - t\pi)(t-1)y'' + 2ty' - (t-1)y = t^2 - t$$

yellow exam #7

$$t(t-\pi)(t-1)\,y'' + 2t\,y' - (t-1)y = t(t-1)$$

$$y'' + \frac{2}{(t-\pi)(t-1)}y' - \frac{1}{t(t-\pi)}y = \frac{1}{t-\pi}$$

problems at $t=0$, $t=1$, $t=\pi$

so intervals are $\boxed{(-\infty,0),\ (0,1),\ (1,\pi) \text{ and } (\pi,\infty)}$

3. (10 points) Find the general solution to the Cauchy-Euler equation

$$t^2y'' + 7ty' + 9y = 0 \qquad t > 0$$

yellow exam #4

$x = \ln t$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{1}{t}\frac{dy}{dx}\right] = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2}\frac{dx}{dt} = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}$$

$$t^2\left[-\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}\right] + 7t\left[\frac{1}{t}\frac{dy}{dx}\right] + 9y = 0$$

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$r = -3$

repeated root

Homogeneous solns

$e^{-3x}, \ xe^{-3x}$

$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

$\to$ translate back to t:

$y(t) = c_1 e^{-3\ln t} + c_2 \ln t \, e^{-3\ln t}$

$$\boxed{y(t) = c_1 t^{-3} + c_2 t^{-3}\ln t \quad t > 0}$$

(3 points) Find the particular solution to the above Cauchy-Euler equation with initial conditions

$$y(1) = 1 \qquad y'(1) = 2$$

and identify the interval on which this solution is defined.

yellow exam #4

$y(t) = c_1 t^{-3} + c_2 t^{-3}\ln t$

$y'(t) = -3c_1 t^{-4} - 3c_2 t^{-4}\ln t + c_2 t^{-4}$

$y(1) = 1 = c_1$

$y'(1) = 2 = -3c_1 + c_2 \Rightarrow c_2 = 5$

$$\boxed{y(t) = t^{-3} + 5t^{-3}\ln t}$$

4. (20 points) Using your general solution to the Cauchy-Euler equation found in the previous problem, use variation of parameters to solve the initial value problem

$$t^2y'' + 7ty' + 9y = t^{-3} \qquad y(1) = 1, \quad y'(1) = 1$$

yellow exam #5

Do not forget to put this ODE into the proper form for variation of parameters!

Clearly identify the particular solution to the ODE, the general solution to the ODE, and the final solution to the IVP.

ODE: $y'' + \frac{7}{t}y' + \frac{9}{t^2} = t^{-5}$

Homogeneous solutions

$y_1(t) = t^{-3}$ $\qquad y_2(t) = t^{-3}\ln t$ $\qquad W = \det\begin{bmatrix} t^{-3} & t^{-3}\ln t \\ -3t^{-4} & -3t^{-4}\ln t + t^{-4}\end{bmatrix}$

$= -3t^{-7}\ln t + t^{-7} + 3t^{-7}\ln t$

$= t^{-7}$

Look for solution $uy_1 + vy_2 = y_P$

$u = \int -\frac{t^{-3}(\ln t)\,t^{-5}}{t^{-7}}\,dt = -\int \frac{\ln t}{t}\,dt = -\frac{1}{2}(\ln t)^2$

$u = \ln t \quad du = \frac{1}{t}dt$

$v = \int \frac{t^{-3}t^{-5}}{t^{-7}}\,dt = \int \frac{1}{t}\,dt = \ln t$

$y_p(t) = -\frac{1}{2}t^{-3}(\ln t)^2 + t^{-3}(\ln t)^2 = \boxed{\frac{1}{2}t^{-3}(\ln t)^2}$ particular solution

General solution

$y(t) = c_1t^{-3} + c_2t^{-3}\ln t + \frac{1}{2}t^{-3}(\ln t)^2$

IVP

$y'(t) = -3c_1t^{-4} - 3c_2t^{-4}\ln t + c_2t^{-4} - \frac{3}{2}t^{-4}(\ln t)^2 + t^{-4}\ln t$

$y(1) = 1 = c_1$

$y'(1) = 1 = -3c_1 + c_2$

$1 = -3 + c_2$

$c_2 = 4$

$$\boxed{y(t) = t^{-3} + 4t^{-3}\ln t + \frac{1}{2}t^{-3}(\ln t)^2}$$

IVP solution

yellow exam #1

5. (3 points) **YOU DO NOT HAVE TO SOLVE THIS.** If you were using the method of undetermined coefficients to solve this equation, what is your best guess for the form of the particular solution?

$$y'' - y' + \frac{1}{4}y = -\frac{25}{4}\sin(t) + 7$$

$r^2 - r + 1/4 = 0$

$(r - 1/2) = 0$

Homogeneous soln's

$e^{-t/2}, te^{-t/2}$

Guess:

$A\sin t + B\cos t + C = y_p$

(3 points) **YOU DO NOT HAVE TO SOLVE THIS.** If you were using the method of undetermined coefficients to solve this equation, what is your best guess for the form of the particular solution?

$$y'' + 6y' + 9y = e^{-3t}$$

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$r = -3$

e^{-3t}, te^{-3t}

Homogeneous solns

Guess

$At^2e^{-3t} = y_p$

(3 points) **YOU DO NOT HAVE TO SOLVE THIS.** If you were using the method of undetermined coefficients to solve this equation, what is your best guess for the form of the particular solution?

$$y'' + 9y = \sin(3t)$$

$r^2 + 9 = 0$

$r = \pm 3i$

$\sin 3t, \cos 3t$

Homogeneous solutions

Guess

$At\sin 3t + Bt\cos 3t$

(3 points) **YOU DO NOT HAVE TO SOLVE THIS.** If you were using the method of undetermined coefficients to solve this equation, what is your best guess for the form of the particular solution?

$$y'' - 6y + 9y = t^2$$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

e^{+3t}, te^{3t}

Guess

$At^2 + Bt + C$

yellow exam #2

6. (6 points) Find the Laplace transform of $f(t) = tu(t-2) = tu_2(t)$ by using the integral definition of the Laplace transform and performing the integral.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{\infty} e^{-st} t\,u(t-2)\,dt =$$

$$\int_2^{\infty} t\,e^{-st}\,dt = -\frac{t}{s} e^{-st}\Big]_2^{\infty} + \frac{1}{s}\int_2^{\infty} e^{-st}\,dt$$

$u = t$ $\quad dv = e^{-st}dt$

$du = dt$ $\quad v = -\frac{1}{s}e^{-st}$

$$= +\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-st}\Big]_2^{\infty} = -\frac{2}{s} e^{-2s} + \frac{1}{s^2} e^{-2s}$$

$$F(s) = e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

(6 points) Find the Laplace transform of

$$f(t) = t\sin(at)$$

Hint: Laplace transform table.

$$\mathcal{L}\{t f(t)\} = (-1)\frac{d}{ds}F(s)$$

$$\mathcal{L}\{t\sin(at)\} = -1\,\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -1\,\frac{d}{ds}\left[a(s^2+a^2)^{-1}\right]$$

$$= a(s^2+a^2)^{-2}(2s)$$

$$= \boxed{\frac{2sa}{(s^2+a^2)^2} = \mathcal{L}\{t\sin t\}}$$

yellow exam #3

7. (8 points) Find the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 4s + 13}{s^3 + 4s^2 + 13s}$$

Partial Fractions:

$$\frac{2s^2+4s+13}{s\underbrace{(s^2+4s+13)}_{\text{not factorable}}} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$$

$$2s^2+4s+13 = As^2+4As+13A + Bs^2+Cs$$

$$13 = 13A \Rightarrow \underline{A=1}$$

$$2s^2 = As^2+Bs^2$$

$$2s^2 = s^2+Bs^2$$

$$\underline{B=1}$$

$$4s = 4As+Cs$$

$$= 4s+Cs$$

$$\Rightarrow \underline{C=0}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{(1)s}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{(s+2)}{(s+2)^2+3^2} - \frac{2}{(s+2)^2+3^2}\right\}$$

complete square

$$= \boxed{1 + e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t}$$

8. (9 points) Find the Laplace transform of the IVP

$$y'' - y = \begin{cases} 0 & \text{if } t < 1 \\ t & \text{if } t \geq 1 \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$$

and solve for $Y(s)$, the Laplace transform of $y(t)$.

$$y'' - y = t\,u(t-1)$$

+1

$$t = f(t-1)$$
$$(t-1)+1 = f(t-1)$$
$$f(t) = t+1$$

+2 +2 +2

$$s^2 Y - Y = e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$Y(s^2-1) = e^{-s}\left[\frac{s+1}{s^2}\right]$$

$$\boxed{Y(s) = e^{-s}\left[\frac{s+1}{s^2(s^2-1)}\right] = e^{-s}\left[\frac{1}{s^2(s-1)}\right]}$$

+2

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \cancel{\mathcal{L}^{-1}\left\{e^{-s}\left[\frac{A}{s} + \frac{B}{s-1}\right]\right\}}$$

$\cancel{As - A + Bs = 1}$, $\cancel{A = -1}$, $\cancel{B = 1}$

$$= \cancel{\mathcal{L}^{-1}\left\{e^{-s}\left[\frac{-1}{s} + \frac{1}{s-1}\right]\right\}} = \mathcal{L}^{-1}\left\{e^{-s}\left[\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}\right]\right\}$$

$$= \cancel{u(t-1)\left[-1 + e^{(t-1)}\right]}$$

$$A(s^2 - s) + B(s-1) + Cs^2 = 1$$

$\underline{B = -1}$

$-A + B = 0$ $\qquad$ $A + C = 0$

$\underline{A = -1}$ $\qquad$ $\underline{C = 1}$

$$\mathcal{L}^{-1}\left\{e^{-s}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s-1}\right]\right\}$$

$$= u(t-1)\left[-1 + (t-1) + e^{t-1}\right] = u(t-1)\left[e^{t-1} + t - 2\right]$$

9. (9 points) Find the inverse Laplace transform for $Y(s)$ in the previous problem to solve the ODE. Hint: factor; cancel. You will need partial fractions.

If you could not find $Y(s)$ in the previous problem, or if you cannot take its inverse transform, you may instead attempt take the inverse transform of

$$Y(s) = e^{-2s}\left[\frac{3s^2+6s+31}{(s-1)(s^2+2s+17)}\right]$$

You will need partial fractions.

$$Y(s) = e^{-s}\left[\frac{1}{s(s-1)}\right] \text{ on previous page}$$

$$Y(s) = e^{-2s}\left[\frac{3s^2+6s+31}{(s-1)(s^2+2s+17)}\right] \xrightarrow{\text{partial fractions}}$$

$$\frac{3s^2+6s+31}{(s-1)(s^2+2s+17)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+17} \Rightarrow 3s^2+6s+31 = As^2+2As+17A + Bs^2 - Bs + Cs - C$$

$s=1$ $\quad 3+6+31 = 20A$

$40 = 20A$

$A = 2$

$3s^2 = As^2 + Bs^2$

$\underline{B=1}$

$6s = 4s - s + Cs$

$6s = 3s + Cs$

$3s = Cs$

$\underline{C=3}$

$$\mathcal{L}^{-1}\left\{e^{-2s}\left[\frac{2}{s-1} + \frac{s+3}{(s+1)^2+4^2}\right]\right\} = \mathcal{L}^{-1}\left\{\left[\frac{2}{s-1} + \frac{s+1}{(s+1)^2+4^2} + \frac{1}{2}\,\frac{4}{(s+1)^2+4^2}\right]e^{-2s}\right\}$$

$$= \underline{u(t-2)\left[2e^{t-2} + e^{-(t-2)}\cos[4(t-2)] + \frac{1}{2}e^{-(t-2)}\sin[4(t-2)]\right]}$$