Homework # 10. Due 11/24/09.

In this assignment, we deal with functions defined on the interval $[0, L]$ and use the inner product

$$< f, g >= \int_0^L f(x)g(x)\,dx.$$

Let L be the differential operator

$$(\mathfrak{L}f)(x) = -f''(x).$$

Problem 1. Let $\psi_k(x) = \sin(\pi kx/L)$.

(a) Show that $< \psi_j, \psi_k >= 0$ when $j \neq i$. This shows that $\{\psi_j\}$ is an orthogonal set with respect to $< \cdot, \cdot >$.

(b) Compute $< \psi_j, \psi_j >$.

Problem 2. Let f and g be in $C^2[0, L]$ and satisfy

$$0 = f(0) = f(L) = g(0) = g(L)$$

.

(a) Use integration by parts to show that

$$< \mathfrak{L}f, g >= \int_0^L f'(x)g'(x)\,dx. \tag{2.1}$$

(b) Use integration by parts applied to the right hand side of (2.1) to show that $< \mathfrak{L}f, g >=< f, \mathfrak{L}g >$. Such an operator (along with the given boundary conditions) is called symmetric.

Problem 3. (a) Use Part b of the previous problem to conclude that $< \psi_k, \psi_j >= 0$ (hint: compute $< \mathfrak{L}\psi_k, \psi_j >$ two different ways).

(b) Show that the eigenfunctions are also orthogonal in the inner product

$$< f, g >_{\mathfrak{L}}= \int_0^L (\mathfrak{L}f)(x)g(x)\,dx.$$

(Hint: use the eigenfunction equation).