Homework #2. Due Feb 2.

In the derivation of the weak form, one is generally allowed to assume that the test function is smooth. The reason for this will be discussed later in the course. Consider the boundary value problem associated with the inhomogeneous box of Lecture 2. Namely: (for $f \neq 0$ defined on $(-1, 1)$)

(1) $-k_1 u'' = f$ on $(-1, 0)

(2) $-k_2 u'' = f$ on $(0, 1)$

(3) $u \text{ continuous at } 0$

(4) $k_1 u_x(0) = k_2 u_x(1)$ Continuity of the flux at
The material interface,

\[ u(-1) = u(1) = 0 \]

(homogeneous boundary conditions).

**Problem 1:** (Derive the weak form).

- Assume that \( \varphi(x) \in C^0([-1, 1]) \) and satisfies \( \varphi(-1) = \varphi(1) \).

Multiply (1) by \( \varphi \) and integrate over \((-1, 1)\) and (2) by \( \varphi \) and integrate over \((0, 1)\) and sum to get one equation with two integrals on the left hand side (LHS).

- Integrate by parts on each integral on the LHS, moving one derivative of \( u \) in each.
(3) use (3) - (5) to simplify the end point terms (coming from the integration by parts).

Problem 2: Let $a = x_0 < x_1 < \ldots < x_N = b$ and set $h_i = x_{i+1} - x_i$, $i = 0, \ldots, N-1$.

The finite element basis function for an interior $DOE x_i$ for piecewise linear (Example of lecture 4) looks like

![Graph](attachment:image.png)

$\phi_i(x)$ vanishes outside of $(x_{i-1}, x_i)$.
a) Derive expressions for $\Phi_i(x)$ on each of the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ for $\Phi_{i+1}(x)$ on $[x_i, x_{i+1}]$ and for $\Phi_{i-1}(x)$ on $[x_{i-1}, x_i]$.

b) Now using the above expressions, compute

$$\int_a^b \Phi_i \Phi_{i-1} \, dx,$$  
$$\int_a^b (\Phi_i(x))^2 \, dx,$$  
$$\int_a^b \Phi_i \Phi_{i+1} \, dx.$$  
$$\int_a^b \Phi_i \Phi_{i+1} \, dx \text{ and } \int_a^b \Phi_i \Phi_{i-1} \, dx.$$. 
Hint: The above integrals are computed by computing the integrals over the subintervals where the product is non-zero and summing. For example, the support properties of \( \phi_i \) and \( \phi_{i+1} \), imply that
\[
\int_{a}^{b} \phi_i \phi_{i+1} = \int_{x_i}^{x_{i+1}} \phi_i(x) \phi_{i+1}(x) \, dx.
\]
Only the integrals involving \( \phi_i \) alone involve two subintervals, i.e.
\[
\int_{a}^{b} \phi_i^2 \, dx = \int_{x_i}^{x_{i+1}} \phi_i(x)^2 \, dx + \int_{x_{i+1}}^{x_i} \phi_i(x)^2 \, dx.
\]