In the first part of this week’s reading, we will prove Theorem 2 of the previous class. We will use the Jordan Decomposition Theorem. To this end, we make the following definition:

**Definition 1.** A Jordan Block is a square matrix of the form

\[
J_{ij} = \begin{cases} 
\lambda & \text{if } i = j, \\
1 & \text{if } j = i + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

The following theorem can be found in any reasonably good text on linear algebra. For more details, see also, http://mathworld.wolfram.com/JordanCanonicalForm.html.

**Theorem 1.** (Jordan Canonical Form Theorem) Any square (complex) matrix \(A\) is similar to a block diagonal matrix with Jordan Blocks on the “block” diagonal, i.e., \(A = NJN^{-1}\) with \(N\) a \(n \times n\) non-singular matrix and \(J\) block diagonal with Jordan blocks on the diagonal.

Here is an example of a block diagonal matrix with Jordan Blocks on the diagonal:

\[
\begin{pmatrix}
J_1 & 0 & 0 \\
0 & J_2 & 0 \\
0 & 0 & J_3
\end{pmatrix}.
\]

Here \(J_i\) are Jordan Blocks of dimension \(k_i \times k_i\) with \(\lambda_i\) on the diagonal. The remaining zero blocks get their sizes from the diagonal blocks, e.g., the 1,3 block has dimension \(k_1 \times k_3\).

**Remark 1.** The above theorem states that given a square complex matrix \(A\), we can write \(J = N^{-1}AN\) where \(J\) is a block diagonal matrix with Jordan Blocks on the diagonal. The similarity matrix \(N\) is, in general, complex even when \(A\) has all real entries. Since similarity transformations preserve eigenvalues and the eigenvalues of an upper triangular matrix (a matrix with zeroes below the diagonal) are the values on the diagonal, the values of \(\lambda\) appearing in the Jordan Blocks are the eigenvalues of the matrix \(A\). As the eigenvalues of matrices with real coefficients are often complex, the similarity matrices end up being complex as well.

We shall use Proposition 1 of reading assignment 2 which we restate here as a reminder.
**Proposition 1.** Let $B$ be an $m \times m$ matrix with possibly complex entries then

$$\|B\|_\infty = \max_{j=1}^m \left( \sum_{k=1}^m |B_{jk}| \right)$$

**Proof of Theorem 2 of Assignment 3.** We start by applying the Jordan Canonical Form Theorem and conclude that $J = N^{-1}AN$ with $J$ being a block diagonal matrix with $K$ Jordan Blocks on the diagonal. The $l$'th block of $J$ has the following form:

$$J^l_{ij} = \begin{cases} 
\lambda_l & \text{if } i = j, \\
1 & \text{if } j = i + 1, \\
0 & \text{otherwise.}
\end{cases}$$

As $\{\lambda_l\}$ for $l = 1, \ldots, K$ are the eigenvalues of $A,$

$$\rho(A) = \max_{l=1}^K |\lambda_l|.$$ 

Let $M$ denote the diagonal matrix with entries $M_{ii} = e^i$ and set

$$\|x\|_* = \|M^{-1}N^{-1}x\|_\infty.$$ 

That this is a norm follows from Problem 2 of Class 2. Now

$$\|A\|_* = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Ax\|_*}{\|x\|_*} = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|M^{-1}N^{-1}Ax\|_\infty}{\|M^{-1}N^{-1}x\|_\infty} = \sup_{y \in \mathbb{C}^n, y \neq 0} \frac{\|M^{-1}N^{-1}ANMy\|_\infty}{\|y\|_\infty} = \|M^{-1}JM\|_\infty.$$ 

The second to last equality above followed from substituting $y = M^{-1}N^{-1}x$ and noting that as $x$ goes over all nonzero vectors, so does $y.$ A direct computation gives

$$(M^{-1}JM)_{ij} = \begin{cases} 
\lambda_l & \text{if } i = j, \\
0 & \epsilon \text{ if } j = i + 1, \\
0 & \text{otherwise.}
\end{cases}$$

Applying the proposition gives,

$$\|A\|_* = \|M^{-1}JM\|_\infty \leq \max_{i=1}^K |\lambda_i| + \epsilon = \rho(A) + \epsilon.$$ 

This completes the proof.
We can get a slightly better version of the corollary of the previous class in the case when \( G \) has real entries.

**Corollary 1.** Let \( G \) be the reduction matrix for a linear iterative method with real entries. Then the iterative method converges for any starting iterate with real values and any right hand side with real values if and only if
\[
\rho(G) < 1.
\]

**Proof.** The proof in the case of \( \rho(G) < 1 \) is identical to that of the previous corollary.

The argument in the case when \( \rho(G) \geq 1 \) needs to be changed since the eigenvector \( \phi \) used in the proof of the previous corollary may be complex. Assume that \( G \) is an \( n \times n \) matrix and \( \rho(G) \geq 1 \) and use the Jordan form theorem to write \( G = N^{-1}JN \). In this case, there is a Jordan Block (of dimension \( k_l \times k_l \)) with diagonal entry \( \lambda_l \) satisfying \( |\lambda_l| \geq 1 \). Suppose that this block corresponds to the indices \( j - k_l + 1, \ldots, j \).

There is a vector \( \phi \in \mathbb{R}^n \) with \( |(N\phi)_j| \neq 0 \). We prove this by contradiction. Suppose \( (N\phi)_j = 0 \) for all \( \phi \in \mathbb{R}^n \) then, by linearity, \( (N(\phi_r + i\phi_i))_j = 0 \) for all \( \phi_r, \phi_i \in \mathbb{R}^n \) (here \( i = \sqrt{-1} \)). This implies that \( (N\phi)_j = 0 \) for all \( \phi \in \mathbb{C}^n \). This cannot be the case since \( N \) is nonsingular and its range is all of \( \mathbb{C}^n \).

Examining the structure of \( J \), it is easy to check that \( (J^n\phi)_j = \lambda_l^i(N\phi)_j \) and hence \( G^n\phi = N^{-1}J^nN\phi \) does not converge to zero as \( i \) tends to infinity.

\[ \square \]

**The Successive Over Relaxation Method.**

In the remainder of this class, we consider the “Successive Over Relaxation Method” (SOR). This is another example of a splitting method. In this case we split \( A = (\omega^{-1}D + L) + ((1 - \omega^{-1})D + U) \), i.e.,
\[
(\omega^{-1}D + L)x = ((\omega^{-1} - 1)D - U)x + b
\]
and obtain the iterative method
\[
(\omega^{-1}D + L)x_{i+1} = ((\omega^{-1} - 1)D - U)x_i + b.
\]
As in the Gauss-Seidel and Jacobi iterations, we can only apply this method to matrices with non-vanishing diagonal entries.

In the above method, \( \omega \) is an iteration parameter which we shall have to choose. When \( \omega = 1 \), the method reduces to Gauss-Seidel. Like Gauss-Seidel, this method can be implemented as a sweep.

\[ ^1 \text{This would be obvious if we knew that } N \text{ were real but this is not the case in general} \]
We first develop a necessary condition on \( \omega \) required for convergence of the SOR method. A simple computation shows that the reduction matrix for the SOR method is

\[
G_{SOR} = (\omega^{-1} D + L)^{-1}((\omega^{-1} - 1)D - U).
\]

From the corollary of the previous class, a necessary condition for SOR to converge for all starting iterates and right hand sides is that \( \rho(G_{SOR}) < 1 \).

The eigenvalues of \( G_{SOR} \) are the roots of the characteristic polynomial,

\[
P(\lambda) = \det(G_{SOR} - \lambda I) = (-1)^n \lambda^n + C_{n-1} \lambda^{n-1} + \cdots + C_0.
\]

Note that \( C_0 = P(0) = \det(G_{SOR}) \) and that

\[
P(\lambda) = (-1)^n \prod_{i=1}^n (\lambda - \lambda_i)
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the roots of \( P \). Expanding the above expression shows that

\[
C_0 = \prod_{i=1}^n \lambda_i = \det(G_{SOR}).
\]

Note that if \( |C_0| \geq 1 \), there has to be at least one root whose absolute value is at least one, i.e., \( \rho(G_{SOR}) \geq 1 \). Consequently, a necessary condition for \( \rho(G_{SOR}) < 1 \) and the convergence of the SOR iteration is that

\[
|C_0| = |\det(G_{SOR})| < 1.
\]

Now, the determinant of a triangular matrix is the product of the entries on the diagonal. Using this and other simple properties of the determinant gives

\[
\det(G_{SOR}) = \frac{\det((\omega^{-1} - 1)D - U)}{\det(\omega^{-1}D + L)} = \frac{\prod_{i=1}^n (\omega^{-1} - 1)D_{i,i}}{\prod_{i=1}^n \omega^{-1}D_{i,i}} = (1 - \omega)^n.
\]

For this product to be less than one in absolute value, it is necessary that \( |1 - \omega| < 1 \), i.e., \( 0 < \omega < 2 \). We restate this as a proposition.

**Proposition 2.** A necessary condition for the SOR method to converge for any starting vector and right hand side is that \( 0 < \omega < 2 \).

We shall provide a convergence theorem for the SOR method. Note that the reduction matrix \( G_{SOR} \) is generally not symmetric even when \( A \) is symmetric. Accordingly, even if \( A \) has real entries, any analysis of \( G_{SOR} \) will have to involve complex numbers as, in general, \( G_{SOR} \) will have complex eigenvalues. Accordingly, we shall provide an analysis for complex Hermitian matrices \( A \).
Definition 2. The conjugate transpose $N^*$ of a general $n \times m$ matrix $N$ with complex entries is the $m \times n$ matrix with entries

$$(N^*)_{i,j} = \bar{N}_{j,i}.$$ 

Here the bar denotes complex conjugate. An $n \times n$ matrix $A$ with complex entries is called Hermitian (or conjugate symmetric) if $A^* = A$.

We shall use the Hermitian inner product $(\cdot, \cdot)$ on $\mathbb{C}^n \times \mathbb{C}^n$ defined by

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i.$$ 

We shall discuss more general Hermitian inner products in later classes. We note that

$$(x, y) = \overline{(y, x)}, \quad (\alpha x, y) = \alpha (x, y), \quad \text{and} \quad (x, \alpha y) = \bar{\alpha} (x, y)$$

for all $x, y \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. In addition, it is easy to see that when $N$ is an $n \times m$ matrix, $N^*$ is the unique matrix satisfying (check it by applying it to the standard basis vectors!)

$$(Nx, y) = (x, N^*y) \quad \text{for all} \quad x \in \mathbb{C}^m, \ y \in \mathbb{C}^n.$$ 

We also note the following properties of a Hermitian $n \times n$ matrix $A$.

- $(Ax, x)$ is real since
  $$\bar{(Ax, x)} = (x, \overline{Ax}) = (Ax, x).$$
- If $A$ is positive definite\(^2\) then $A_{i,i} > 0$ since $A_{i,i} = ( Ae_i, e_i ) > 0$ where $e_i$ denotes the $i$'th standard basis vector for $\mathbb{C}^n$.

We finish this class by stating a convergence theorem for SOR. Its proof will be given in the next class.

Theorem 2. Let $A$ be an $n \times n$ Hermitian positive definite matrix and $\omega$ be in $(0, 2)$. Then the SOR method for iteratively solving $Ax = b$ converges for any starting vector and right hand side.

\(^2\)A complex $n \times n$ matrix $A$ is positive definite if $(Ax, x) > 0$ for all non zero $x \in \mathbb{C}^n$. 