ANALYSIS OF THE SPECTRUM OF A CARTESIAN PERFECTLY MATCHED LAYER (PML) APPROXIMATION TO ACOUSTIC SCATTERING PROBLEMS

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Abstract. In this paper, we study the spectrum of the operator which results when the Perfectly Matched Layer (PML) is applied in Cartesian geometry to the Laplacian on an unbounded domain. This is often thought of as a complex change of variables or "complex stretching." The reason that such an operator is of interest is that it can be used to provide a very effective domain truncation approach for approximating acoustic scattering problems posed on unbounded domains. Stretching associated with polar or spherical geometry lead to constant coefficient operators outside of a bounded transition layer and so even though they are on unbounded domains, they (and their numerical approximations) can be analyzed by more standard compact perturbation arguments. In contrast, operators associated with Cartesian stretching are non-constant in unbounded regions and hence cannot be analyzed via a compact perturbation approach. Alternatively, to show that the scattering problem PML operator associated with Cartesian geometry is stable for real nonzero wave numbers, we show that the essential spectrum of the higher order part only intersects the real axis at the origin. This enables us to conclude stability of the PML scattering problem from a uniqueness result given in a subsequent publication.

1. Introduction.

In this paper, we study the spectrum of an operator which results from a complex stretching (PML) of the Laplace operator on an unbounded domain. The PML technique can be applied to acoustic scattering problems and gives rise to an effective domain truncation strategy. The complex stretching can be thought of as highly absorbing fictitious layer which attenuates outgoing radiation [3, 6]. In fact, in many cases, this leads to a new problem, still on the unbounded domain, which preserves the solution inside the layer while decaying rapidly outside [5, 6, 13]. Because of this decay, it is feasible to develop numerical approximations by domain truncation and the application of the finite element method. Although the solution of the truncated problem no longer coincides with the original inside the layer, it often can be shown to be exponentially close.

Such stretched operators have been studied by other authors when the transformation was based on polar or spherical coordinates [5, 16, 6] and coordinates associated with a smooth convex surface [14]. We note that a complete analysis of the discrete problem involves stability of the infinite and truncated domain source problems at the continuous level and the analysis of the truncated finite element

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approximation. The last step in the case of acoustic scattering is classical once the stability of the continuous truncated problem has been verified (see, e.g., [20, 21]).

We consider the exterior Helmholtz problem with Sommerfeld radiation condition,

$$\begin{aligned} -\Delta u - k^2 u &= 0 \text{ in } \bar{\Omega}^c, \\ u &= g \text{ on } \partial\Omega, \\ \lim_{r\to\infty} r^{1/2}\left|\frac{\partial u}{\partial r} - iku\right| &= 0. \end{aligned}$$

Here k is real and non-negative and Ω is a bounded domain with Lipschitz continuous boundary contained in the square† $[-a,a]^2$ for some positive a.

The simplest example of a Cartesian PML approximation involves an even function $\tilde{\sigma}$ satisfying

$$\begin{aligned} &\tilde{\sigma}(x) = 0 \text{ for } |x| \le a, \\ &\tilde{\sigma}(x): \text{ increasing for } a < x < b, \\ &\tilde{\sigma}(x) = \sigma_0 \text{ for } |x| \ge b. \end{aligned} \tag{1.1}$$

Here $0 < a < b$ and $\sigma_0 > 0$ is a parameter (the PML strength). The PML reformulation leads to the study of a source problem: for $f \in L^2(\bar{\Omega}^c)$, find $\hat{u} \in H^1_0(\bar{\Omega}^c)$ satisfying

$$A(\hat{u},\phi) - k^2(d(x_1)d(x_2)\hat{u},\phi) = (d(x_1)d(x_2)f,\phi) \text{ for all } \phi \in H^1_0(\bar{\Omega}^c). \tag{1.2}$$

Here $H^1_0(\bar{\Omega}^c)$ denotes the Sobolev space of order one on $\bar{\Omega}^c$ consisting of complex valued functions which vanish on $\partial\Omega$, $d(x) = 1 + i(x\tilde{\sigma}(x))'$ and

$$\begin{aligned} A(u,v) &= \int_{\Omega^c}\left[\frac{d(x_2)}{d(x_1)}\frac{\partial u}{\partial x_1}\frac{\partial \bar{v}}{\partial x_1} + \frac{d(x_1)}{d(x_2)}\frac{\partial u}{\partial x_2}\frac{\partial \bar{v}}{\partial x_2}\right] d\mathbf{x}, \\ (f,g) &= \int_{\Omega^c} f\bar{g}\, d\mathbf{x}. \end{aligned} \tag{1.3}$$

In [5], an analysis of the source problem on the infinite domain with spherical PML was given by first showing that the resulting form was coercive up to a lower order perturbation on a bounded domain. A standard argument by compact perturbation [17, 23] then shows stability of the source problem once uniqueness has been established. Unfortunately, this perturbation approach fails for Cartesian PML. The problem is, e.g., that the coefficient of the x_1 derivatives in the form on the left hand side of (1.2) equals $-k^{-2}$ times that of the zeroth order term when $x_1 \in (-a,a)$, i.e., when $d(x_1) = 1$. As $\bar{\Omega}^c \cap ((-a,a)\times\mathbb{R})$ is an unbounded domain, we cannot restore coercivity by a zeroth order perturbation on a BOUNDED domain.

We need to circumvent the compact perturbation approach. We do this by analyzing the essential spectrum of the unbounded operator $\widetilde{L} : H^{-1}(\bar{\Omega}^c) \to H^{-1}(\bar{\Omega}^c)$ with domain $H^1_0(\bar{\Omega}^c)$ defined for $v \in H^1_0(\bar{\Omega}^c)$ by $\widetilde{L}v = f$, where $f \in H^{-1}(\bar{\Omega}^c)$ is given by

$$< f, \bar{d}(x_1)\,\bar{d}(x_2)\phi >= A(v,\phi) \text{ for all } \phi \in H^1_0(\bar{\Omega}^c). \tag{1.4}$$

Here $< \cdot,\cdot >$ denotes the duality pairing. As usual, if $f \in L^2(\bar{\Omega}^c)$, then the duality pairing coincides with the L^2-inner product.

†We consider a domain in $\mathbb{R}^2$ for convenience. The extension to domains in $\mathbb{R}^3$ is completely analogous with small restriction. See Remark 2.3.

We take the definition of essential spectrum $\sigma_{ess}(\widetilde{L})$ to be the set of points in the spectrum (the complement of the resolvent $\rho(\widetilde{L})$) excluding those in the discrete spectrum $\sigma_d(\widetilde{L})$ (isolated points of the spectrum with finite algebraic multiplicity). There are other notions of essential spectrum, some of which are discussed in [8].

We shall see that $\widetilde{L}$ is a (well defined) closed unbounded operator on $H^{-1}(\bar{\Omega}^c)$ with domain $H_0^1(\bar{\Omega}^c)$ provided that $\tilde{\sigma}$ is smooth enough. Note that $\widetilde{L}$ is a weak form of the operator

$$-\widetilde{\Delta} \equiv -\frac{1}{d(x_1)}\frac{\partial}{\partial x_1}\left(\frac{1}{d(x_1)}\frac{\partial}{\partial x_1}\right) - \frac{1}{d(x_2)}\frac{\partial}{\partial x_2}\left(\frac{1}{d(x_2)}\frac{\partial}{\partial x_2}\right).$$

The major result of this paper is the identification of the essential spectrum $\sigma_{ess}(\widetilde{L})$ (see Figure 1.) and the conclusion that $\sigma_{ess}(\widetilde{L})$ intersects the real axis only at the origin. This means that the only way that k^2 (for real k with $k \neq 0$) can fail to be in the resolvent set for $\widetilde{L}$ is that there is an eigenvector of $\widetilde{L}$ associated with k^2. In a subsequent paper analyzing the truncated Cartesian PML approximation [12], we show that the sesquilinear form in (1.2) (and its adjoint) satisfies a uniqueness result of the form: If $u \in H_0^1(\bar{\Omega}^c)$ satisfies

$$A(u,v) - k^2(d(x_1)d(x_2)u, v) = 0 \text{ for all } v \in H_0^1(\bar{\Omega}^c),$$

then $u = 0$. This uniqueness result prohibits eigenvectors with eigenvalue k^2 and combining it with the results of this paper shows that, in fact, k^2 is in the resolvent set of $\widetilde{L}$ for any real nonzero k. This conclusion implies the "inf-sup" conditions for the variational problem (1.2) and leads to existence, uniqueness and stability of its solution (for suitable f). Using this in [12], we show stability of the truncated problem as well as its finite element approximation.

We note that considering $\widetilde{L}$ as an unbounded operator on $H^{-1}(\bar{\Omega}^c)$ (in contrast to $L^2(\bar{\Omega}^c)$ as often done, e.g., [4, 9]) enables us to deal with domains with only Lipschitz continuous boundaries. This results in an operator which is not local. Nevertheless the operator $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$ still satisfies some classical properties associated with unbounded operators on $L^2(\mathbb{R}^2)$, e.g., the boundary of the essential spectrum of $\widetilde{L}$ depends only on its behavior at infinity and hence coincides with that of the natural extended operator (which we still denote by $\widetilde{L}$) on all of $H^{-1}(\mathbb{R}^2)$. The extended operator $\widetilde{L}$ on $H^{-1}(\mathbb{R}^2)$ is related to an extended operator $-\widetilde{\Delta}$ on $L^2(\mathbb{R}^2)$ and we show that their spectra coincide. As $-\widetilde{\Delta}$ is a tensor product operator, its spectrum as an operator on $L^2(\mathbb{R}^2)$ can be identified from the spectrum of its one dimension components using classical results.

The study of the spectral structure of elliptic operators is interesting in its own right and has a long history. In particular, spectral theory for Schrödinger operators and scattering problems posed on exterior domains has been extensively investigated in, e.g., [10, 15, 18]. Interestingly, spectral deformation theory developed by Aguilar, Balslev, Combes and Simon [1, 2, 22] is intimately related to spherical PML operators and its use for the computation of resonances. One result of this theory shows how the essential spectrum of the original selfadjoint operator is moved by complex coordinate stretching based on spherical geometry. The present paper develops this transformation for the case of Cartesian based stretching.

The outline of the remainder of the paper is as follows. In Section 2, we give some preliminaries and state some tools for identifying the boundary of the essential spectrum of operators from their behavior at infinity. In Section 3, we study the spectrum of the one dimensional PML operator. These results are used in Section 4 to identify the essential spectrum of the operator $-\widetilde{\Delta}$ defined on $L^2(\mathbb{R}^2)$ and subsequently that of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$.

2. Preliminary tools.

We give some preliminary results and tools for the analysis of the spectrum of operators in this section. We state a remark concerning a slightly more general PML formulation.

Remark 2.1. In the introduction, we considered a simple PML example where the same stretching function was used in each direction. In an application where the domain more naturally fits into a rectangle $[-a_1, a_1] \times [-a_2, a_2]$, it is more reasonable (and computationally efficient) to use direction dependent PML stretching functions. For example, we use even functions $\tilde{\sigma}_j$, for $j = 1, 2$ satisfying (1.1) with a, b and σ_0 replaced by a_j, b_j and σ_{j0}, respectively. The only changes in (1.2) and (1.4) involve replacement of $d(x_j)$ by $d_j(x_j) \equiv 1+i(x_j\tilde{\sigma}_j(x_j))'$. As the analysis presented below is identical for direction dependent PML stretching, for convenience of notation, from here on, we shall revert back to the case of the introduction, i.e., $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}$.

Remark 2.2. We further assume that the PML function $\tilde{\sigma}$ is in $C^2(\mathbb{R})$. This will be sufficient to guarantee that the unbounded operators discussed in the introduction are well defined and closed.

We next show that $\widetilde{L}$ is well defined. Indeed, for $v \in H_0^1(\bar{\Omega}^c)$,

$$|A(v, \phi)| \le C^{\ddagger} \|v\|_{H^1(\bar{\Omega}^c)} \|\phi\|_{H^1(\bar{\Omega}^c)} \text{ for all } \phi \in H_0^1(\bar{\Omega}^c).$$

As multiplication by a bounded C^1 function whose absolute value is bounded away from zero gives an isomorphism of $H_0^1(\bar{\Omega}^c)$ onto $H_0^1(\bar{\Omega}^c)$, it follows that

$$|A(v, (\bar{d}(x_1)\,\bar{d}(x_2))^{-1}\phi)| \le C\|v\|_{H^1(\bar{\Omega}^c)} \|\phi\|_{H^1(\bar{\Omega}^c)}$$

so there is a unique $f \in H^{-1}(\bar{\Omega}^c)$ satisfying (1.4) and $\widetilde{L}$ is well defined. Moreover,

$$\|\widetilde{L}v\|_{H^{-1}(\bar{\Omega}^c)} \le C\|v\|_{H^1(\bar{\Omega}^c)} \text{ for all } v \in H_0^1(\bar{\Omega}^c). \tag{2.1}$$

From the definition (1.1) of $\tilde{\sigma}(x)$, it follows that $0 \le (x\tilde{\sigma}(x))' \le \sigma_M$ for some σ_M, which shows that $d(x) = 1 + i(x\tilde{\sigma}(x))'$ is in the set $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) = 1, 0 \le \mathrm{Im}(z) \le \sigma_M\}$. Now, if $\theta = \arg(1 + i\sigma_M)$, then it is easy to show that

$$\mathrm{Re}(d(x)/d(y)) \ge \alpha \text{ and } \mathrm{Re}(e^{-i\theta}d(x)d(y)) \ge \alpha \text{ for all } x, y \in \mathbb{R}$$

with $\alpha = 1/(1 + \sigma_M^2)$. This implies that for $z_0 = -e^{-i\theta}$,

$$|A(u, u) - z_0(d(x_1)d(x_2)u, u)| \ge \alpha\|u\|^2_{H^1(\bar{\Omega}^c)} \text{ for all } u \in H_0^1(\bar{\Omega}^c). \tag{2.2}$$

Remark 2.3. In the case of $\mathbb{R}^3$, the Cartesian PML sesquilinear form $A(\cdot, \cdot)$ analogous to (1.3) has $d(x_k)d(x_l)/d(x_j)$ as the coefficient of the x_j derivative term for mutually different $j, k, l = 1, 2, 3$. In contrast to the case of $\mathbb{R}^2$, a restriction is needed for coercivity. Specifically, we require $d(x)$ to satisfy $\arg(1 + i\sigma_M) < \pi/3$. Then coercivity can be obtained examining the behavior of $d(x_k)d(x_l)/d(x_j)$ and $e^{-i\theta}d(x_j)d(x_k)d(x_l)$ for appropriately chosen θ.

‡Here and in the remainder of the paper, C denotes a generic positive constant which may take on different values in different places often depending on the spectral parameter (z or z_0).

This, and the discussion above, implies that given $f \in H^{-1}(\bar{\Omega}^c)$, there is a unique $u \in H^1_0(\bar{\Omega}^c)$ satisfying

$$A(u,\phi) - z_0(d(x_1)d(x_2)u,\phi) = < f, \bar{d}(x_1)\,\bar{d}(x_2)\phi > \text{ for all } \phi \in H^1_0(\bar{\Omega}^c). \tag{2.3}$$

Moreover,

$$\|u\|_{H^1(\bar{\Omega}^c)} \le C\|f\|_{H^{-1}(\bar{\Omega}^c)}. \tag{2.4}$$

It is immediate that $(\widetilde{L} - z_0 I)u = f$ and so z_0 is in the the resolvent set $\rho(\widetilde{L})$. This implies that the operator $\widetilde{L}$ is closed, its resolvent set is non-empty and its spectrum is well defined.

As alluded to in the introduction, we define an extended operator (still denoted by $\widetilde{L}$) defined for $v \in H^1(\mathbb{R}^2)$ by $\widetilde{L}v = f$, where $f \in H^{-1}(\mathbb{R}^2)$ is defined by

$$< f, \bar{d}(x_1)\,\bar{d}(x_2)\phi > = A(v,\phi) \text{ for all } \phi \in H^1(\mathbb{R}^2). \tag{2.5}$$

Clearly, $d(x)$ is well defined for all $x \in \mathbb{R}$ and (2.5) makes sense. For $f \in L^2(\mathbb{R}^2)$ the duality pairing is the integral $(\cdot,\cdot)_{\mathbb{R}^2}$.

The argument above shows that $z_0 \in \rho(\widetilde{L})$ for the extended operator and so $\widetilde{L}$ is closed, its resolvent set is non-empty, and its spectrum is well defined.

To develop the same properties for $-\widetilde{\Delta}$ as an operator on $L^2(\mathbb{R}^2)$ with domain $H^2(\mathbb{R}^2)$, elliptic regularity comes into play. Specifically since $\tilde{\sigma}$ is $C^2(\mathbb{R})$, classical arguments involving difference quotients (see, also, [5, 19]) can be used to show that when $f \in L^2(\mathbb{R}^2)$, the solution u of the extended version of (2.3) is in $H^2(\mathbb{R}^2)$ and satisfies

$$\|u\|_{H^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}. \tag{2.6}$$

This means that u is in the domain of $-\widetilde{\Delta}$ and satisfies

$$(-\widetilde{\Delta} - z_0 I)u = f,$$

i.e., $z_0 \in \rho(-\widetilde{\Delta})$. This immediately gives the desired results as above.

In this paper, we study the spectrum of $-\widetilde{\Delta}$ on $L^2(\mathbb{R}^2)$ and $\widetilde{L}$ on $H^{-1}(\mathbb{R}^2)$ to describe the essential spectrum of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$. As a first step, we have the following theorem.

Theorem 2.4. *The spectrum of $\widetilde{L}$ as an unbounded operator on $H^{-1}(\mathbb{R}^2)$ (with domain $H^1(\mathbb{R}^2)$) is the same as the spectrum of $-\widetilde{\Delta}$ on $L^2(\mathbb{R}^2)$ (with domain $H^2(\mathbb{R}^2)$).*

Before proving the theorem, we observe the following lemma.

Lemma 2.5. *The point z is in $\rho(\widetilde{L})$ (as an operator on $H^{-1}(\mathbb{R}^2)$) if and only if the following two inf-sup conditions hold: For all u in $H^1(\mathbb{R}^2)$,*

$$\|u\|_{H^1(\mathbb{R}^2)} \le C \sup_{\phi \in H^1(\mathbb{R}^2)} \frac{|A_z(u,\phi)|}{\|\phi\|_{H^1(\mathbb{R}^2)}}, \tag{2.7}$$

and

$$\|u\|_{H^1(\mathbb{R}^2)} \le C \sup_{\phi \in H^1(\mathbb{R}^2)} \frac{|A_z(\phi,u)|}{\|\phi\|_{H^1(\mathbb{R}^2)}}, \tag{2.8}$$

where $A_z(\cdot,\cdot) \equiv A(\cdot,\cdot) - z(d(x_1)d(x_2)\cdot,\cdot)_{\mathbb{R}^2}$.

Proof. The inf-sup conditions immediately imply that the map: $\widetilde{L} - zI : H^1(\mathbb{R}^2) \to H^{-1}(\mathbb{R}^2)$ is an isomorphism. This means that if the inf-sup conditions hold for z, then z is in the resolvent set $\rho(\widetilde{L})$.

We already know from (2.2) that the inf-sup conditions hold for z_0. It suffices to prove the first inf-sup condition as the second follows from it since the coefficients of $A_z(\cdot,\cdot)$ are complex symmetric (but not Hermitian).

Suppose that z is in $\rho(\widetilde{L})$. To prove (2.7), let u be in $C_0^\infty(\mathbb{R}^2)$ and $v \in H^1(\mathbb{R}^2)$ be the unique function satisfying (cf., (2.2))

$$A_{z_0}(v,\phi) = A_z(u,\phi) \text{ for all } \phi \in H^1(\mathbb{R}^2).$$

Setting $u_0 = u - v$, a simple computation gives

$$A_z(u_0,\phi) = (z - z_0)(d(x_1)d(x_2)v, \phi)_{\mathbb{R}^2} \text{ for all } \phi \in H^1(\mathbb{R}^2)$$

or

$$(\widetilde{L} - zI)u_0 = (z - z_0)v. \tag{2.9}$$

Since $z \in \rho(\widetilde{L})$,

$$\|u_0\|_{H^{-1}(\mathbb{R}^2)} \le C\|v\|_{H^{-1}(\mathbb{R}^2)}. \tag{2.10}$$

Also,

$$A_{z_0}(u_0,\phi) = (z - z_0)(d(x_1)d(x_2)[v + u_0], \phi)_{\mathbb{R}^2} \text{ for all } \phi \in H^1(\mathbb{R}^2)$$

and hence by (2.4) and (2.10)

$$\|u_0\|_{H^1(\mathbb{R}^2)} \le C\|v\|_{H^{-1}(\mathbb{R}^2)}.$$

Thus, using (2.2) gives

$$\begin{aligned}\|u\|_{H^1(\mathbb{R}^2)} &\le \|v\|_{H^1(\mathbb{R}^2)} + \|u_0\|_{H^1(\mathbb{R}^2)} \le C\|v\|_{H^1(\mathbb{R}^2)} \\ &\le C \sup_{\phi \in H^1(\mathbb{R}^2)} \frac{|A_{z_0}(v,\phi)|}{\|\phi\|_{H^1(\mathbb{R}^2)}} = C \sup_{\phi \in H^1(\mathbb{R}^2)} \frac{|A_z(u,\phi)|}{\|\phi\|_{H^1(\mathbb{R}^2)}}.\end{aligned}$$

This proves (2.7) and completes the proof of the lemma. □

Remark 2.6. The lemma holds for $\widetilde{L}$ defined on $H^{-1}(\bar{\Omega}^c)$ with the inf-sup conditions involving the supremum over $H_0^1(\bar{\Omega}^c)$. The proof is identical.

Corollary 2.7. *If z in $\rho(-\widetilde{\Delta})$ (as an operator on $L^2(\mathbb{R}^2)$), then* (2.7) *and* (2.8) *hold for z and hence $z \in \rho(\widetilde{L})$ on $H^{-1}(\mathbb{R}^2)$.*

Proof. The proof that $z \in \rho(-\widetilde{\Delta})$ implies (2.7) and (2.8) is essentially identical to that of the lemma except that (2.10) is replaced by

$$\|u_0\|_{L^2(\mathbb{R}^2)} \le C\|v\|_{L^2(\mathbb{R}^2)}. \tag{2.11}$$

□

Remark 2.8. Let Ω_δ denote the square domain $[-\delta,\delta]^2$ with $\delta \ge b$. We fix $z \in \rho(-\widetilde{\Delta})$ (as an operator on $L^2(\mathbb{R}^2)$). For the analysis in [12], we shall require that the inf-sup conditions of Lemma 2.5 still hold with $H^1(\mathbb{R}^2)$ replaced by $H_0^1(\Omega_\delta)$ uniformly for $\delta > \delta_0 = \delta_0(z)$. Examining the proof of the above lemma, we see that for this to hold it suffices to show that for $\delta > \delta_0$, $z \in \rho(\widetilde{\Delta}_\delta)$ (as an operator on $L^2(\Omega_\delta)$ with domain $H^2(\Omega_\delta)\cap H_0^1(\Omega_\delta)$) and there is a constant C depending only on δ_0 and z satisfying

$$\|(-\widetilde{\Delta}_\delta - zI)^{-1}\|_{L^2(\Omega_\delta)} \le C \tag{2.12}$$

for all $\delta > \delta_0$. The existence of δ_0 and C will be verified in the proof of Theorem 4.8.

Proof of Theorem 2.4. That $\rho(-\widetilde{\Delta})$ is contained in $\rho(\widetilde{L})$ is given by the above corollary. The other direction, $\rho(\widetilde{L}) \subseteq \rho(-\widetilde{\Delta})$, follows from Lemma 2.5, the two inf-sup conditions and elliptic regularity (the argument is identical that used earlier in this section to show $z_0 \in \rho(-\widetilde{\Delta})$). □

To connect the spectrum of the extended operators to that of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$, we require the concepts of local compactness of operators, the Weyl spectrum and the Zhislin spectrum. Let $\mathcal{U}$ be $\bar{\Omega}^c$ or $\mathbb{R}^m$ for $m = 1, 2$.

Definition 2.9. For $B \subset \mathcal{U}$, let χ_B denote the characteristic function on B. If a closed operator T with $\rho(T) \neq \emptyset$ satisfies the condition that $\chi_B(T-\lambda I)^{-1}$ is compact for any bounded open set $B \subset \mathcal{U}$ and for some $\lambda \in \rho(T)$ (and so any $\lambda \in \rho(T)$), then T is called *locally compact*.

Remark 2.10. The local compactness of operators defined on $L^2(\mathcal{U})$ is standard. We consider the case of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$ as that of $\widetilde{L}$ on $H^{-1}(\mathbb{R}^2)$ is identical. Because of the inf-sup conditions, when $z \in \rho(\widetilde{L})$, $u_n = (\widetilde{L} - zI)^{-1} f_n$ satisfies (2.4). Letting B be as above and $\|f_n\|_{H^1(\bar{\Omega}^c)} = 1$, it follows that there is a subsequence of $\{\chi_B u_n\}$ which converges in $L^2(B)$. Convergence in $L^2(B)$, in turn, implies convergence in $H^{-1}(\bar{\Omega}^c)$.

Definition 2.11. Let T be a closed operator on a Hilbert space $\mathcal{H}$. A *Weyl sequence* $\{u_n\}$ for T and $\lambda \in \mathbb{C}$ is a sequence such that $\|u_n\|_{\mathcal{H}} = 1$, $u_n \to 0$ weakly and $\|(T-\lambda I)u_n\|_{\mathcal{H}} \to 0$. The set of all λ such that a Weyl sequence exists for T and λ is called the *Weyl spectrum* $W(T)$ of T.

The Weyl spectrum $W(T)$ of a closed operator T is related to the essential spectrum $\sigma_{ess}(T)$ of T as follows.

Theorem 2.12. [7, Theorem 3.1] *Let T be a closed operator on a Hilbert space $\mathcal{H}$ with $\rho(T) \neq \emptyset$. Then $W(T) \subset \sigma_{ess}(T)$ and the boundary of $\sigma_{ess}(T)$ is contained in $W(T)$. Finally, $W(T) = \sigma_{ess}(T)$ if and only if each connected component of the complement of $W(T)$ contains a point of $\rho(T)$.*

Definition 2.13. Let T be a closed operator on $\mathcal{H} \equiv H^{-1}(\mathcal{U})$ or $L^2(\mathbb{R}^m)$ for $m = 1, 2$. A *Zhislin sequence* u_n for T and $\lambda \in \mathbb{C}$ is a sequence such that $\|u_n\|_{\mathcal{H}} = 1$, $\mathrm{supp}(u_n) \cap K = \emptyset$ for each compact set $K \subset \mathcal{U}$ and for all n large, and such that $\|(T-\lambda I)u_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. The set of all λ such that a Zhislin sequence exists for T and λ is called the *Zhislin spectrum* $Z(T)$ of T.

Since every Zhislin sequence converges to zero weakly, it is obvious that $Z(T) \subset W(T)$. In general, these two sets are not necessarily equal but sometimes they coincide as shown in the following theorems.

Theorem 2.14. *Let T be a locally compact, closed operator on $L^2(\mathbb{R}^m)$ such that $\rho(T) \neq \emptyset$ and $C_0^\infty(\mathbb{R}^m)$ is a core. Let $\chi \in C_0^\infty(\mathbb{R}^m)$ be such that $\chi|_{B(0,r)} = 1$ for some $r > 0$, where $B(0,r)$ is a ball centered at the origin and of radius r. We define*

$\chi_n(x) \equiv \chi(x/n)$. Suppose that there exists $\varepsilon(n)$ such that $\varepsilon(n) \to 0$ as $n \to \infty$, and that for all $u \in C_0^\infty(\mathbb{R}^m)$

$$\|[T, \chi_n]u\|_{L^2(\mathbb{R}^m)} \le \varepsilon(n)(\|Tu\|_{L^2(\mathbb{R}^m)} + \|u\|_{L^2(\mathbb{R}^m)}). \tag{2.13}$$

Here $[T, \chi_n]$ is the commutator of T and χ_n: $[T, \chi_n]u = T(\chi_n u) - \chi_n T u$ for $u \in C_0^\infty(\mathbb{R}^m)$. Then $Z(T) = W(T)$.

This result for operators on $L^2(\mathbb{R}^m)$ is given in [7, Theorem 3.2]. We note that $C_0^\infty(\bar{\Omega}^c)$ is still a core of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$ and we have a similar theorem. Its proof is essentially the same as that of Theorem 2.14 in [7].

Theorem 2.15. *Let T be a locally compact, closed operator on $H^{-1}(\mathcal{U})$ with domain $H_0^1(\mathcal{U})$ such that $\rho(T) \neq \emptyset$. Let χ_n be as in the previous theorem. Suppose that there exists $\varepsilon(n)$ such that $\varepsilon(n) \to 0$ as $n \to \infty$, and that for all $u \in H_0^1(\mathcal{U})$*

$$\|[T, \chi_n]u\|_{H^{-1}(\mathcal{U})} \le \varepsilon(n)(\|Tu\|_{H^{-1}(\mathcal{U})} + \|u\|_{H^{-1}(\mathcal{U})}). \tag{2.14}$$

Then $Z(T) = W(T)$.

3. Spectrum of the one dimensional PML operator on $L^2(\mathbb{R})$.

In this section, we consider the spectrum of the one dimensional stretched operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ defined by

$$\widetilde{\mathcal{D}} = -\frac{1}{d(x)} \frac{\partial}{\partial x} \left(\frac{1}{d(x)} \frac{\partial}{\partial x} \right).$$

A weak form corresponding to $\widetilde{\mathcal{D}} u = f$ for $f \in L^2(\mathbb{R})$ is given by: find $u \in H^1(\mathbb{R})$ satisfying

$$a(u, v) = (d(x) f, v)_{\mathbb{R}} \text{ for all } v \in H^1(\mathbb{R}),$$

where

$$a(u, v) = \left(\frac{1}{d(x)} u', v' \right)_{\mathbb{R}} \text{ for all } u, v \in H^1(\mathbb{R}).$$

The arguments showing that $\widetilde{\mathcal{D}}$ is well defined as an operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ are identical to those given in Section 2 for $-\widetilde{\Delta}$. In fact, z_0 is in $\rho(\widetilde{\mathcal{D}})$. Additional properties are given in the following lemma.

Lemma 3.1. *The operator $\widetilde{\mathcal{D}}$ on $L^2(\mathbb{R})$ is locally compact and satisfies* (2.13).

Proof. The local compactness of $\widetilde{\mathcal{D}}$ immediately follows from the compact embedding of $H^2(B)$ as a subset of $L^2(B)$ for bounded B (we take $\lambda = z_0 \in \rho(\widetilde{\mathcal{D}})$).

It remains to show that $\widetilde{\mathcal{D}}$ satisfies (2.13). As in Section 2, for $u \in C_0^\infty(\mathbb{R})$,

$$\|u'\|^2_{L^2(\mathbb{R})} \le \alpha^{-1} \mathrm{Re}(d(x)^{-1} u', u')_{\mathbb{R}} = \alpha^{-1} \mathrm{Re}(\widetilde{\mathcal{D}} u, \bar{d}(x) u)_{\mathbb{R}}.$$

Thus,

$$\|u'\|_{L^2(\mathbb{R})} \le C(\|\widetilde{\mathcal{D}} u\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}). \tag{3.1}$$

Expanding $[\widetilde{\mathcal{D}}, \chi_n]u$ and noting that all terms cancel except those involving differentiation of χ_n gives

$$\|[\widetilde{\mathcal{D}}, \chi_n]u\|_{L^2(\mathbb{R})} \le C(\|\chi_n'' u\|_{L^2(\mathbb{R})} + \|\chi_n' u'\|_{L^2(\mathbb{R})} + \|\chi_n' u\|_{L^2(\mathbb{R})}).$$

Since $\|\chi_n'\|_\infty, \|\chi_n''\|_\infty \le C/n$ for large n, by (3.1),

$$\begin{aligned}\|[\widetilde{\mathcal{D}}, \chi_n]u\|_{L^2(\mathbb{R})} &\le \frac{C}{n}(\|u'\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})})\\ &\le \frac{C}{n}(\|\widetilde{\mathcal{D}}u\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}),\end{aligned}$$

which completes the proof. □

Proposition 3.2. *Let $\widetilde{\mathcal{D}}$ be as above. Then*

$$\sigma(\widetilde{\mathcal{D}}) = \sigma_{ess}(\widetilde{\mathcal{D}}) = \{z \in \mathbb{C} \ : \ \arg(z) = -2\arg(1 + i\sigma_0)\}.$$

Proof. Let $S \equiv -(1+i\sigma_0)^{-2}\partial^2/\partial x^2$ be defined on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. Note that S coincides with $\widetilde{\mathcal{D}}$ for $x \notin [-b, b]$. Lemma 3.1 holds for S so

$$W(S) = Z(S) = Z(\widetilde{\mathcal{D}}) = W(\widetilde{\mathcal{D}}) \tag{3.2}$$

by Theorem 2.14. Moreover,

$$\sigma(S) = \sigma_{ess}(S) = \{z \in \mathbb{C} \ : \ \arg(z) = -2\arg(1 + i\sigma_0)\} = W(S), \tag{3.3}$$

where the last equality followed from Theorem 2.12. Applying Theorem 2.12 to $\widetilde{\mathcal{D}}$ and using (3.2) shows that $\sigma_{ess}(\widetilde{\mathcal{D}})$ is also given by (3.3).

To complete the proof, we will show that the discrete spectrum of $\widetilde{\mathcal{D}}$ is empty. Indeed, if λ is in the discrete spectrum of $\widetilde{\mathcal{D}}$, then there is an eigenvector $u \in H^2(\mathbb{R})$ such that $\widetilde{\mathcal{D}}u = \lambda u$. It is easy to see that

$$u(x) = C_1 e^{i\sqrt{\lambda}x(1+i\tilde{\sigma}(x))} + C_2 e^{-i\sqrt{\lambda}x(1+i\tilde{\sigma}(x))}. \tag{3.4}$$

For $x \notin [-b, b]$,

$$u(x) = C_1 e^{i\sqrt{\lambda}x(1+i\sigma_0)} + C_2 e^{-i\sqrt{\lambda}x(1+i\sigma_0)}.$$

Examining this expression, it is clear that the only way that u can be in $L^2(\mathbb{R})$ is that $C_1 = C_2 = 0$, i.e., $u = 0$. This completes the proof of the lemma. □

4. The spectrum of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$.

We prove the main theorem concerning the essential spectrum of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$ in this section. We start by examining the spectrum of $-\widetilde{\Delta}$ on $L^2(\mathbb{R}^2)$. We first consider the tensor product operator associated with components coming from the one dimensional operator $\widetilde{\mathcal{D}}$, specifically

$$\widetilde{\mathcal{T}} = \widetilde{\mathcal{D}} \otimes I + I \otimes \widetilde{\mathcal{D}}. \tag{4.1}$$

This operator is defined on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^2)$ with domain $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$. We note that $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ is dense in $H^2(\mathbb{R}^2)$ and that $\widetilde{\mathcal{T}}$ coincides with $-\widetilde{\Delta}$ on $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$. This means that $-\widetilde{\Delta}$ is the closure of $\widetilde{\mathcal{T}}$.

To characterize the spectrum of $-\widetilde{\Delta}$, we introduce the following theorem on tensor product operators.

Theorem 4.1. [18, Theorem XIII.35] *Let $\mathcal{A}$ and $\mathcal{B}$ be the generators of bounded holomorphic semigroups on a Hilbert space $\mathcal{H}$. Let dom$(\mathcal{A})$ and dom$(\mathcal{B})$ be the domains of $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{H}$, respectively. If $\mathcal{C}$ is the closure of the operator $\mathcal{A}\otimes I + I\otimes\mathcal{B}$ defined on dom$(\mathcal{A}) \otimes$ dom$(\mathcal{B})$, then $\mathcal{C}$ generates a bounded holomorphic semigroup and*

$$\sigma(\mathcal{C}) = \sigma(\mathcal{A}) + \sigma(\mathcal{B}).$$

The next theorem provides a criterion for an operator to be a generator of a holomorphic semigroup. First, the following definition is required.

Definition 4.2. Let T be a closed operator on a Hilbert space $\mathcal{H}$. T is called m-sectorial with a vertex at $z=0$ and a semi-angle $\delta \in [0,\pi/2)$ if the numerical range of T, $\mathcal{N}(T) = \{(Tu,u) \in \mathbb{C} \ : \ u \in dom(T)\}$, is contained in a sector $S_\delta = \{z \in \mathbb{C} \ : \ |\arg(z)| \le \delta\}$ and $(\mathbb{C} \setminus S_\delta) \cap \rho(T) \neq \emptyset$.

Theorem 4.3. [11, IX Theorem 1.24] *Let T be an m-sectorial operator on a Hilbert space $\mathcal{H}$. Then T generates a bounded holomorphic semigroup.*

Lemma 4.4. *There exist a real and positive constant β and a complex constant η such that $T \equiv \eta\widetilde{\mathcal{D}} + \beta I$ is m-sectorial.*

Proof. The spectrum of T is a line from β to infinity and hence $(\mathbb{C} \setminus S_\delta) \cap \rho(T) \neq \emptyset$ for any $\delta \in [0,\pi/2)$.

Let $\eta = 1 + i\sigma_M$, where $\sigma_M = \max_{t\in\mathbb{R}}\{\sigma(t)\}$. It suffices to show that for a positive β, there exists a positive constant C such that $\mathrm{Re}(Tu,u)_{\mathbb{R}} \ge C|\mathrm{Im}(Tu,u)_{\mathbb{R}}|$ for all $u \in H^2(\mathbb{R})$ since this implies that the numerical range $\mathcal{N}(T)$ of T is contained in the sector S_δ with a vertex at $z=0$ and a semi-angle $\delta = \tan^{-1}(1/C)$. Now, for $u \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}(Tu,u)_{\mathbb{R}} &= -\int_{\mathbb{R}} \frac{\eta}{d(x)} \frac{\partial}{\partial x}\left(\frac{1}{d(x)}\frac{\partial u}{\partial x}\right)\bar{u}\ dx + \beta\|u\|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \frac{\eta}{d(x)^2}\left|\frac{\partial u}{\partial x}\right|^2 dx + \int_{\mathbb{R}} \frac{\eta}{d(x)}\left(\frac{1}{d(x)}\right)' \frac{\partial u}{\partial x}\bar{u}\ dx + \beta\|u\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{4.2}$$

Note that there exist positive constants c_1 and c_2 such that

$$\mathrm{Re}\left(\frac{\eta}{d(x)^2}\right) \ge c_1 \quad \text{and} \quad \left|\frac{\eta}{d(x)}\left(\frac{1}{d(x)}\right)'\right| \le c_2. \tag{4.3}$$

Using (4.3), applying the Schwarz inequality and the arithmetic-geometric mean inequality gives that for any positive γ,

$$\begin{aligned}\mathrm{Re}(Tu,u)_{\mathbb{R}} &\ge c_1\|u'\|^2_{L^2(\mathbb{R})} + \beta\|u\|^2_{L^2(\mathbb{R})} - \frac{c_2}{2}(\gamma\|u'\|^2_{L^2(\mathbb{R})} + 1/\gamma\|u\|^2_{L^2(\mathbb{R})}) \\ &= (c_1 - \gamma c_2/2)\|u'\|^2_{L^2(\mathbb{R})} + (\beta - c_2/(2\gamma))\|u\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{4.4}$$

Choosing γ small enough and β large enough implies

$$\mathrm{Re}(Tu,u)_{\mathbb{R}} \ge C_R\|u\|^2_{H^1(\mathbb{R})}.$$

On the other hand, it easily follows that

$$|\mathrm{Im}(Tu,u)_{\mathbb{R}}| \le C_I\|u\|^2_{H^1(\mathbb{R})}. \tag{4.5}$$

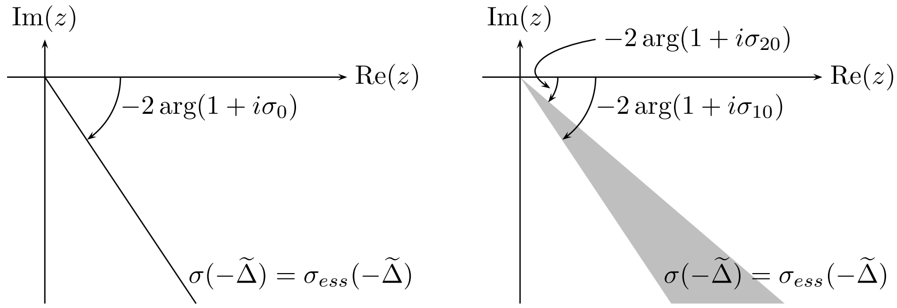
Combining these results and noting that $C_0^\infty(\mathbb{R})$ is dense in $H^2(\mathbb{R})$ finishes the proof of the lemma. □

Combining the above results gives the following theorem concerning the spectrum of $-\widetilde{\Delta}$, which we state for the more general PML formulation discussed in Remark 2.1. Let

$$\mathcal{S} \equiv \{z \in \mathbb{C} \ : \ -2\arg(1+i\sigma_{20}) \le \arg(z) \le -2\arg(1+i\sigma_{10})\}$$

when $\sigma_{10} \le \sigma_{20}$ and

$$\mathcal{S} \equiv \{z \in \mathbb{C} \ : \ -2\arg(1+i\sigma_{10}) \le \arg(z) \le -2\arg(1+i\sigma_{20})\}$$



(a) The case of $\sigma_{10} = \sigma_{20} = \sigma_0$.

(b) The case when $\sigma_{10} \neq \sigma_{20}$.

FIGURE 1. The essential spectrum of $-\widetilde{\Delta}$ on $L^2(\mathbb{R}^2)$ (which coincides with that of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$).

when $\sigma_{10} > \sigma_{20}$.

Theorem 4.5. *The spectrum of $-\widetilde{\Delta}$ on $L^2(\mathbb{R}^2)$ with domain $H^2(\mathbb{R}^2)$ is given by*

$$\sigma(-\widetilde{\Delta}) = \sigma_{ess}(-\widetilde{\Delta}) = \mathcal{S} \tag{4.6}$$

(see Figure 1.).

Proof. We first consider the case when $\sigma_{10} = \sigma_{20} = \sigma_0$. Since $\eta\widetilde{\mathcal{D}} + \beta I$ is m-sectorial, it follows from Theorem 4.3 that $\eta\widetilde{\mathcal{D}}+\beta I$ generates a bounded holomorphic semigroup. By Theorem 4.1

$$\sigma(-\eta\widetilde{\Delta} + 2\beta I) = \sigma(\eta\widetilde{\mathcal{D}} + \beta I) + \sigma(\eta\widetilde{\mathcal{D}} + \beta I) = \sigma(\eta\widetilde{\mathcal{D}} + 2\beta I).$$

Translating by -2β and multiplying by $1/\eta$ gives

$$\sigma(-\widetilde{\Delta}) = \sigma(\widetilde{\mathcal{D}}).$$

In the case when $\sigma_{10} \neq \sigma_{20}$, $\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2$ are $\widetilde{\mathcal{D}}$ defined with $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, respectively for each component. As above, we have

$$\sigma(-\widetilde{\Delta}) = \sigma(\widetilde{\mathcal{D}}_1) + \sigma(\widetilde{\mathcal{D}}_2) = \mathcal{S}.$$

This completes the proof of the theorem. □

We are now in a position to state and prove the main result of this paper.

Theorem 4.6. *The essential spectrum of $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$ with domain $H^1_0(\bar{\Omega}^c)$ is contained in $\mathcal{S}$.*

Proof. The spectrum of $-\widetilde{\Delta}$ on $L^2(\mathbb{R}^2)$ is the same as $\widetilde{L}$ on $H^{-1}(\mathbb{R}^2)$ by Theorem 2.4. Clearly, both $\widetilde{L}$ on $H^{-1}(\mathbb{R}^2)$ and $\widetilde{L}$ on $H^{-1}(\bar{\Omega}^c)$ are locally compact. To finish the proof of the theorem, it suffices to show that they satisfy (2.14). Indeed, in that case, we apply Theorem 2.15 to conclude that

$$\begin{aligned}\mathcal{S} \supseteq W(\widetilde{L})(\text{on } H^{-1}(\mathbb{R}^2)) &= Z(\widetilde{L})(\text{on } H^{-1}(\mathbb{R}^2)) \\ &= Z(\widetilde{L})(\text{on } H^{-1}(\bar{\Omega}^c)) = W(\widetilde{L})(\text{on } H^{-1}(\bar{\Omega}^c)).\end{aligned}$$

The theorem follows from Theorem 2.12 since $W(\widetilde{L})$(on $H^{-1}(\bar{\Omega}^c)$) contains the boundary of $\sigma_{ess}(\widetilde{L})$ (on $H^{-1}(\bar{\Omega}^c)$).

We verify (2.14) in the case of $H^{-1}(\bar{\Omega}^c)$. The other case is essentially identical. For χ_n defined in Theorem 2.15 and $u \in H_0^1(\bar{\Omega}^c)$, a simple computation shows that for $\phi \in C_0^\infty(\bar{\Omega}^c)$,

$$\begin{aligned} < [\widetilde{L}, \chi_n]u, \bar{d}(x_1)\,\bar{d}(x_2)\phi > &= A(\chi_n u, \phi) - A(u, \bar{\chi}_n \phi) \\ &= \left(\frac{d(x_2)}{d(x_1)}\frac{\partial \chi_n}{\partial x_1}u, \frac{\partial \phi}{\partial x_1}\right) + \left(\frac{d(x_1)}{d(x_2)}\frac{\partial \chi_n}{\partial x_2}u, \frac{\partial \phi}{\partial x_2}\right) \\ &\quad - \left(\frac{d(x_2)}{d(x_1)}\frac{\partial \chi_n}{\partial x_1}\frac{\partial u}{\partial x_1}, \phi\right) - \left(\frac{d(x_1)}{d(x_2)}\frac{\partial \chi_n}{\partial x_2}\frac{\partial u}{\partial x_2}, \phi\right). \end{aligned}$$

Using the fact that the first derivatives of χ_n can be bounded by C/n gives

$$| < [\widetilde{L}, \chi_n]u, \bar{d}(x_1)\,\bar{d}(x_2)\phi > | \le \frac{C}{n}\|u\|_{H^1(\bar{\Omega}^c)}\|\phi\|_{H^1(\bar{\Omega}^c)}.$$

Now

$$\|u\|_{H^1(\bar{\Omega}^c)} \le C\|(\widetilde{L} - z_0 I)u\|_{H^{-1}(\bar{\Omega}^c)} \le C(\|\widetilde{L}u\|_{H^{-1}(\bar{\Omega}^c)} + \|u\|_{H^{-1}(\bar{\Omega}^c)}). \tag{4.7}$$

Combining the above results shows that

$$| < [\widetilde{L}, \chi_n]u, \bar{d}(x_1)\,\bar{d}(x_2)\phi > | \le \frac{C}{n}(\|\widetilde{L}u\|_{H^{-1}(\bar{\Omega}^c)} + \|u\|_{H^{-1}(\bar{\Omega}^c)})\|\phi\|_{H^1(\bar{\Omega}^c)}.$$

The desired result (2.14) follows as in the proof of (2.1). This completes the proof of the theorem. □

Remark 4.7. By cutting down functions of the form

$$f(x, y) = e^{i[\gamma x/(1+i\sigma_{10}) + \beta y/(1+i\sigma_{20})]}$$

with γ and β positive, it is possible to show that

$$\gamma^2/(1 + i\sigma_{10})^2 + \beta^2/(1 + i\sigma_{20})^2 \in Z(\widetilde{L}).$$

As any point of $\mathcal{S}$ can be obtained this way, $\sigma_{ess}(\widetilde{L})$ (on $H^{-1}(\bar{\Omega}^c)$) equals $\mathcal{S}$.

The last result of this paper provides uniform inf-sup conditions for the truncated problem.

Theorem 4.8. *Let z be in $\rho(-\widetilde{\Delta})$. Then there is a δ_0 such that for all $\delta > \delta_0$ and u in $H_0^1(\Omega_\delta)$,*

$$\|u\|_{H_0^1(\Omega_\delta)} \le C \sup_{\phi \in H_0^1(\Omega_\delta)} \frac{|A_z(u, \phi)|}{\|\phi\|_{H_0^1(\Omega_\delta)}}, \tag{4.8}$$

and

$$\|u\|_{H_0^1(\Omega_\delta)} \le C \sup_{\phi \in H_0^1(\Omega_\delta)} \frac{|A_z(\phi, u)|}{\|\phi\|_{H_0^1(\Omega_\delta)}}. \tag{4.9}$$

Proof. Let z be in $\rho(-\widetilde{\Delta})$. As observed in Remark 2.8, it suffices to verify (2.12). If the constants in (2.12) are not uniformly bounded as δ goes to infinity, then there is a sequence $\{(\delta_n, u_n)\}$ satisfying

$$\begin{gathered} u_n \in H^2(\Omega_{\delta_n}) \cap H_0^1(\Omega_{\delta_n}), \qquad \delta_n \to \infty \quad \text{as } n \to \infty, \\ \|(-\widetilde{\Delta} - zI)u_n\|_{L^2(\Omega_{\delta_n})} \le \frac{1}{n}, \qquad \|u_n\|_{L^2(\Omega_{\delta_n})} = 1. \end{gathered}$$

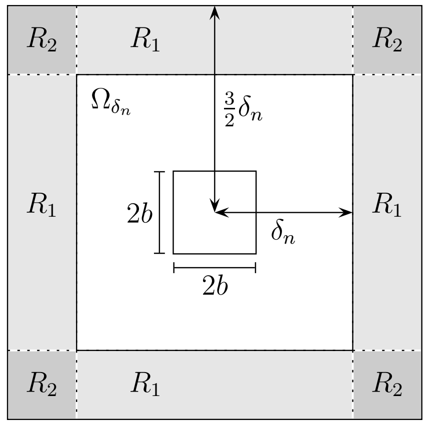


Figure 2. The reflection subdomains.

We assume that $\delta_n \geq 2b$. We next extend u_n to $\Omega_{3\delta_n/2}$ by odd reflection. Specifically, we define the extended function $\tilde{u}_n$ by first doing an odd reflection across $\partial\Omega_{\delta_n}$ into the regions labeled R_1 in Figure 2. Next, we do another odd reflection (across the boundary between R_1 and R_2) from the regions labeled R_1 into those labeled R_2. The values obtained in a R_2 region are independent of the choice of component of R_1 used in the reflection. It is easy to see that the resulting function $\tilde{u}_n$ is in $H^2(\Omega_{3\delta_n/2})$. Moreover, $(-\widetilde{\Delta} - zI)\tilde{u}_n(\tilde{x})$ for any $\tilde{x} \in \Omega_{3\delta_n/2} \setminus \Omega_{\delta_n}$ coincides with $\pm(-\widetilde{\Delta} - zI)u_n(x)$ where x is the point in Ω_{δ_n} which reflects into $\tilde{x}$. Accordingly,

$$\|(-\widetilde{\Delta} - zI)\tilde{u}_n\|_{L^2(\Omega_{3\delta_n/2})} \leq 2\|(-\widetilde{\Delta} - zI)u_n\|_{L^2(\Omega_{\delta_n})} \leq \frac{2}{n}.$$

Let χ be a smooth function on $\mathbb{R}^2$ with values in $[0,1]$ satisfying $\chi(x) = 1$ on $[-1,1]^2$ and $\chi(x) = 0$ outside of $(-3/2, 3/2)^2$. Define $\chi^n(x) = \chi(x/\delta_n)$. We shall show that

$$\|[\widetilde{\Delta}, \chi^n]\tilde{u}_n\|_{L^2(\Omega_{3\delta_n/2})} \leq \frac{C}{n}. \tag{4.10}$$

Note that if (4.10) holds, then $w_n = \chi_n \tilde{u}_n$ is in $H^2(\mathbb{R})$ and satisfies:

$$\|w_n\|_{L^2(\mathbb{R})} \geq \|u_n\|_{L^2(\Omega_{\delta_n})} = 1$$

and

$$\begin{aligned} \|(-\widetilde{\Delta} - zI)w_n\|_{L^2(\mathbb{R})} &\leq \|[\widetilde{\Delta}, \chi^n]\tilde{u}_n\|_{L^2(\Omega_{3\delta_n/2})} \\ &\quad + \|\chi^n(-\widetilde{\Delta} - zI)\tilde{u}_n\|_{L^2(\Omega_{3\delta_n/2})} \leq \frac{C}{n}. \end{aligned}$$

This contradicts the fact that $z \in \rho(-\widetilde{\Delta})$ ($-\widetilde{\Delta}$ as an operator on $L^2(\mathbb{R}^2)$).

To verify (4.10), we first note that by (2.2),

$$\|u_n\|^2_{H^1(\Omega_{\delta_n})} \leq C(\|u_n\|^2_{L^2(\Omega_{\delta_n})} + |A(u_n, u_n)|).$$

Now, u_n is in $H^2(\Omega_{\delta_n}) \cap H^1_0(\Omega_{\delta_n})$ and integration by parts gives

$$|A(u_n, u_n)| = (-\widetilde{\Delta}u_n, \bar{d}(x_1)\,\bar{d}(x_2)u_n)_{\Omega_{\delta_n}} \leq C\|\widetilde{\Delta}u_n\|_{L^2(\Omega_{\delta_n})}\|u_n\|_{L^2(\Omega_{\delta_n})}$$

from which it follows that

$$\|u_n\|_{H^1(\Omega_{\delta_n})} \leq C(\|u_n\|_{L^2(\Omega_{\delta_n})} + \|(-\widetilde{\Delta} - zI)u_n\|_{L^2(\Omega_{\delta_n})}).$$

Because of the reflection construction, this inequality extends to

$$\|\tilde{u}_n\|_{H^1(\Omega_{3\delta_n/2})} \leq 2C(\|u_n\|_{L^2(\Omega_{\delta_n})} + \|(-\widetilde{\Delta} - zI)u_n\|_{L^2(\Omega_{\delta_n})}) \leq C. \tag{4.11}$$

Expanding $[\widetilde{\Delta}, \chi^n]$ gives

$$\begin{aligned}[\widetilde{\Delta}, \chi^n]\tilde{u}_n = &\frac{1}{d(x)}\frac{\partial}{\partial x}\left(\frac{1}{d(x)}\chi_x^n \tilde{u}\right) + \frac{1}{d(x)^2}\chi_x^n \tilde{u}_x \\ &+ \frac{1}{d(y)}\frac{\partial}{\partial y}\left(\frac{1}{d(y)}\chi_y^n \tilde{u}\right) + \frac{1}{d(y)^2}\chi_y^n \tilde{u}_y.\end{aligned} \tag{4.12}$$

We note that $d^{-1}(x)$ and $d'(x)$ are uniformly bounded and $\|\chi_x^n\|_{L^\infty(\mathbb{R}^2)}$, $\|\chi_{xx}^n\|_{L^\infty(\mathbb{R}^2)}$, $\|\chi_y^n\|_{L^\infty(\mathbb{R}^2)}$ and $\|\chi_{yy}^n\|_{L^\infty(\mathbb{R}^2)}$ are all bounded by C/n. Thus (4.10) follows from integrating (4.12), using the above estimates, (4.11) and the triangle inequality. This completes the proof of the theorem. □

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