Math 220H
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Exam 1
October 16$^{\text{th}}$, 2007

Name:____________________
ID#:____________________

Rules of the Game.

(a) Give complete explanations for your answers.

(b) Ask if you have a question.

(c) Never give an answer which trivializes the problem.

The Exam.

Problem 1 *(5 points each) Define*

(a) the converse of the statement $p \Rightarrow q$,

$q \Rightarrow p$

(b) and the contrapositive of $p \Rightarrow q$.

$\neg q \Rightarrow \neg p$

Problem 2 *(15 pts) Prove that for any propositions, p and q,*

$$S: \quad [p \Rightarrow (p \Rightarrow q)] \Rightarrow [p \Rightarrow q]$$

S false iff $\begin{cases} p \Rightarrow (p \Rightarrow q) & T \text{ and} \\ p \Rightarrow q & F \end{cases}$

But $p \Rightarrow q$ F implies p T.

$\therefore$ $p \Rightarrow (p \Rightarrow q)$ F contradicting

$\therefore$ S is true.

Problem 3 *(15 points) Let p, q and r be propositions. Show that* $p \Rightarrow q$ *is equivalent to* $p \wedge \neg q \Rightarrow r \wedge \neg r$.

$(p \Rightarrow q) \Leftrightarrow \neg(p \wedge \neg q)$

Since $r \wedge \neg r$ is always F

$(p \wedge \neg q \Rightarrow r \wedge \neg r) \Leftrightarrow \neg(p \wedge \neg q)$

$\therefore (p \Rightarrow q) \Leftrightarrow (p \wedge \neg q \Rightarrow r \wedge \neg r)$.

Problem 4 *(15 pts) Let U denote the universe and A, B and C be three subsets. Prove that if* $A \cup B = A^c \cup C$, *and* $A \cup C = A^c \cup B$, *then* $B \cup C = U$.

Pick $x \in U$.

Case 1. $x \in A$.

$\therefore x \in A \cup B = A^c \cup C$. Since $x \notin A^c$, we have $x \in C$.

$\therefore x \in B \cup C$.

Case 2. $x \in A^c$. (or $x \notin A$)

$\therefore x \in A^c \cup C = A \cup B$. Since $x \notin A$, we have $x \in C$. $\therefore x \in B \cup C$.

$\therefore$ every $x \in B \cup C$. $\therefore U = B \cup C$.

$= 22(464 - 2(173)) - 15(173)$

$= -59\ (173) + 22(464)$

$= -59\left(637 - 1(464)\right) + 22(464)$

$= 81\ (464) - 59(637)$

$\therefore\ x = -59$ and

$y = 81$

Check

$$\begin{array}{r} 464 \\ 81 \\ \hline 464 \\ 3712 \\ \hline 37584 \end{array} \qquad \begin{array}{r} 637 \\ 59 \\ \hline 5733 \\ 3185 \\ \hline 37583 \end{array}$$

$37{,}584 - 37{,}583 = 1$ ✓

Problem 5 *(10 pts) A club has a rule for new members: each must always tell the truth or always lie. They know who does which. If I meet three of them on the street and they make the statements below, which ones (if any) should I believe?*

L *A says: "All three of us are liars."*

T *B says: "Exactly two of us are liars."*

L *C says: "The other two are liars."*

(1) A tells truth : ∴ A is a liar. $\Rightarrow \bot$. ∴ A is a liar.

(2) C tells truth : ∴ A tells truth, contradicting (1).

∴ C is a liar.

(3) B lies : ∴ there are 0, 1 or 3 liars. Since A, C are liars, all three must be liars ∴ A tells the truth $\Rightarrow \bot$.

∴ $\boxed{\text{B tells the truth}}$

Problem 6 *(10 points)*

(i). Find gcd(637, 464) *and*

(ii). find x and y such that $637x + 464y = \gcd(637, 464)$.

Let S = set of common divisors of

(i) $637 = 1\cdot(464) + 173$ ∴ S = set of common divisors of 464 and 173

$464 = 2(173) + 118$ ∴ S = set of comm. div's of 173 and 118

$173 = 1(118) + 55$ ∴ S = " " " " " 118 and 55.

$118 = 2(55) + 8$ ∴ S = " " " " " 55 and 8

$55 = 6(8) + 7$ ∴ S = " " " " " 8 and 7

$8 = 1\cdot 7 + 1$ ∴ S = " " " " " 7 and 1

which is 1

∴ gcd(637, 464) = 1.

(ii) $1 = 8 - 1\cdot 7 = 8 - 1(55 - 6\cdot 8) = 7\cdot 8 - 1(55) = 7(118 - 2(55)) - 1(55) = -15(55) + 7(118)$

$= -15(173 - 1\cdot 118) + 7(118) = 22(118) - 15(173)$

Problem 7 *(25 points) Prove that the cube root of 24 is irrational.*

Suppose $(24)^{1/3}$ is rational. Then $\exists\ a, b \in \mathbb{N}$, $\gcd(a,b)=1$

s.t. $(24)^{1/3} = \frac{a}{b}$. $\therefore$ $24b^3 = a^3$. $\therefore$ $3 \mid 24b^3 = a^3$.

Claim, $3 \mid a$. If not, since 3 is a prime, $\gcd(3,a)=1$.

$\therefore$ since $3 \mid a \cdot a^2$, $3 \mid a^2 = a \cdot a$. $\therefore$ $3 \mid a$. $\Rightarrow\Leftarrow$.

$\therefore$ $3 \mid a$. $\therefore$ $a = 3c$. $\therefore$ $24b^3 = 3^3 c^3$.

$\therefore$ $3^2 c^3 = 8b^3$. $\therefore$ $3 \mid 3^2 c^3 = 8b^3$. Since $\gcd(3,8)=1$,

$3 \mid b^3$. As above $3 \mid b$. $\therefore$ $3 \mid a$ and $3 \mid b$,

contradicting $\gcd(a,b)=1$.

Hence $(24)^{1/3}$ cannot be rational.

$\therefore$ $(24)^{1/3}$ is irrational.