

Rules of the Game.

- (a) Give complete explanations for your answers.
- (b) Ask if you have a question.
- (c) Never give an answer which trivializes the problem.

The Exam.

1. (a) Prove that for any propositions p, q and r ,

$$(p \Rightarrow q \vee r) \leftrightarrow (p \wedge \neg q \Rightarrow r).$$

- (b) Here is a Theorem that you may use to solve this part of the problem.

Theorem 1 *If $k, l \in \mathbb{N}$ have no common divisors, then there exist $x, y \in \mathbb{Z}$ such that $kx + ly = 1$.*

Use part (a) and the above Theorem to prove:

If p is a prime and $a, b \in \mathbb{N}$ and if p divides the product, ab , then either p divides a or p divides b .

- (c) Prove by induction that if the prime, p divides the product $a_1 a_2 \cdots a_n$, then p divides one of the natural numbers a_1, a_2, \dots, a_n .
2. (a) If A and B are sets in the Universe, S , and $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.
- (b) If A and B are sets in the Universe, S , and $\mathcal{P}(\mathcal{P}(A)) = \mathcal{P}(\mathcal{P}(B))$, then $A = B$.
3. If $A, B, C \subseteq S$ and $(A \cup B) \cap C \subseteq B^c$, then $A \cap B \cap C = \emptyset$.
4. Prove : If $a, b \in \mathbb{N}$ and $a^2 = b^3$, then the following are true:
- (a) if 2 divides a , then 4 divides b ,
 - (b) if 4 divides b , then 8 divides a .
 - (c) Now give an example of $a, b \in \mathbb{N}$ with a even and $a^2 = b^3$, but b is not divisible by 8.
5. Prove that if a is irrational, then $1/a$ is irrational.
6. Prove that the fifth root of 15 is irrational.
7. (a) Let $a_1 = a_2 = 1$ and for $n \geq 3$, $a_n = 2a_{n-1} + 3a_{n-2}$. Prove that $a_n \leq 5^n$.
- (b) Find a better result.