Practice for 220 Final

Problem 1 *Let* p, q *and* r *be statements. Prove that*

$$\big((p \to q) \to r\big) \longleftrightarrow \big((p \vee r) \wedge (q \to r)\big).$$

Problem 2 *Prove:* $39\big|(53^{103} + 103^{53})$.

Problem 3 *Let* A, B *and* C *be subsets of (the universe)* U. *Show that if*

$$A \cup B = B \cap C, \text{ then } A \subseteq B \subseteq C.$$

Problem 4 *Define an relation on the natural numbers by:* xRy *iff the first (from the left) four digits of* x *equals the first four digits of* y. *Is this an equivalence relation?*

Problem 5 *Show that* $\gcd(\gcd(a, b), c) = \gcd(a, \gcd(b, c))$.

Problem 6 *Define* $p \downarrow q$ *to be* $\neg(p \vee q)$. *Show that*

$$p \to q$$

is equivalent to

$$((p \downarrow p) \downarrow q) \downarrow (p \downarrow p) \downarrow q).$$

Problem 7 *Prove that* $(2n)!! \geq (\frac{2}{3}n)^n$, *for all* $n \geq 1$. *You can assume* $(1 + \frac{1}{n})^n \leq 3$.

Problem 8 *Show that if* a, b *and* n *are natural numbers with* $b > a$ *and* $n \geq 2$, *then*

$$(b - a)^n < b^n - a^n.$$

Problem 9 *Prove:* $13\big|(23^{37} + 15^{16})$.

Problem 10 *Show by induction that* $\frac{1}{n+1}\binom{2n}{n} \geq 1$ *for all* $n \geq 1$.

Problem 11 *Let* $a_1 = a_2 = 1$ *and assume that for* $n \geq 3, a_n \leq 5a_{n-1} + 8a_{n-2}$. *Show that* $a_n \leq 7^n$.

Problem 12 *Solve the following equations simultaneously.*

$x \equiv 1$ *(mod 7)* , $x \equiv 3$ *(mod 5)* *and* $x \equiv 5$ *(mod 3)*

Problem 13 *Let* p *be a prime and* a *a natural number. Show that*

$$a^{p^2} \equiv a \ (mod\ p)\ .$$

Problem 14 *If* $10a \equiv 10b$ *(mod 6)* , *show that* $a \equiv b$ *(mod 3)* .

Problem 15 *Define a relation on the natural numbers by: xRy iff the sum of the digits of x equals the sum of the digits of y. Show that this is an equivalence relation.*

Problem 16 *Show that* $\gcd(\gcd(a, b), c) = \gcd(a, \gcd(b, c))$.

Problem 17 *Fix natural numbers a, b, c. Give a description of the smallest natural number in the set*

$$\{ax + by + cz : x, y, z \text{ integers } \}.$$

Problem 18 *Prove the following theorem: There do not exist prime numbers a, b and c and a natural number $n \geq 2$ such that*

$$a^n + b^n = c^n.$$

Here are some hints.

1. *Use a proof by contradiction.*
2. *What happens if both a and b are odd?*

Problem 19 *Show that there are infinitely many primes of the form $8n + 5$ or $8m + 7$. (We don't know if there are infinitely many of each.)*