1. (a) $p \Rightarrow (q \vee r)$ is false iff
p is T and $(q \vee r)$ is F
iff p is T and q and r are F.

$p \wedge \neg q \Rightarrow r$ is false iff
p T and $\neg q$ is T and r is F
iff p T, q F and r F.

Hence, the two statements are false (and, hence, true) at the same time.

(b) Using part (a) we know that

$$p \mid ab \Rightarrow [p \mid a \text{ or } p \mid b]$$

is equivalent to

$$(p \mid ab \wedge p \nmid a) \Rightarrow p \mid b.$$

(We can now assume both $p \mid ab$ and the additional $p \nmid a$. But we have to prove "more", since we must prove $p \mid b$, not just $p \mid a$ or $p \mid b$.)

But, if $p \nmid a$ then p and a have no nontrivial common divisors, since p is a prime. (A common divisor, d, would have to divide both p and a. But the only divisors of p are 1 and p. So, if we want a non-trivial common divisor, $\boxed{d=p}$ $\therefore$ since $d|a$, $p|a$. Recall we're assuming $p \nmid a$.)

We can now apply the Theorem given on the test. So we get $x, y \in \mathbb{Z}$, s.t. $ax + py = 1$. Multiply both sides of the equation by b, we get:

$$abx + pby = b.$$

The left side is divisible by p, since $p | abx$ and $p | pby$. Hence, p must divide b. This is what we wanted.

(c) Proof by induction. We know the result for $n=1$ and $n=2$. Assume true for $n \geq 2$ and show for $n+1$. So we're assuming

$$p \mid b_1 b_2 \cdots b_n \cdot b_{n+1} = \underbrace{(b_1 \cdots b_n)} \cdot \underbrace{b_{n+1}}$$

So, p divides a product of two natural #'s ($b_1 \cdots b_n$ and b_{n+1}).

By the $n=2$ case (proved in (b))

we know that either

$p \mid (b_1 \cdots b_n)$ or $p \mid b_{n+1}$.

But by the inductive hypothesis

if $p \mid b_1 \cdots b_n$, then $p \mid b_1$ or $p \mid b_2$ or $\cdots$ or $p \mid b_n$. So, we have

$p \mid b_1$ or $\cdots$ or $p \mid b_n$ or $p \mid b_{n+1}$ $\boxtimes$.

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2. (a) Suppose $\mathcal{P}(A) = \mathcal{P}(B)$. Then

$A \subseteq A \Rightarrow A \in \mathcal{P}(A) = \mathcal{P}(B)$. $\therefore A \subseteq B$.

Similarly, $B \in \mathcal{P}(B) = \mathcal{P}(A)$. $\therefore B \subseteq A$.

$\therefore A = B$.

(b) $\mathcal{P}(\boxed{\mathcal{P}(A)}) = \mathcal{P}(\boxed{\mathcal{P}(B)}) \Rightarrow \mathcal{P}(A) = \mathcal{P}(B)$

(by applying part (a) to the sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$ in place of A & B)).

Now, again by part (a), ~~$\emptyset$~~ since we now have $\mathcal{P}(A) = \mathcal{P}(B)$, we get $A = B$. $\blacksquare$

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3. ~~P~~ $(A \cup B) \cap C \subseteq B^c$. If $x \in A \cap B \cap C$, then $x \in A \cup B$ (and $\underline{x \in B}$) and $x \in C$.

$\therefore$ $x \in (A \cup B) \cap C \subseteq B^c$ and $\underline{x \in B}$

So, $x \in B^c \cap B = \emptyset$, a contradiction.

$\therefore$ there is no ~~elt~~ x in $A \cap B \cap C$.

$\therefore$ $A \cap B \cap C = \emptyset$.

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4. (a) $2 | a \Rightarrow 4 | a^2 \Rightarrow 4 | b^3 \Rightarrow 2 | b^3 \Rightarrow 2 | b$.

$\therefore$ $b = 2c$ for some $c \in \mathbb{N}$.

$\therefore$ $a^2 = b^3 = (2c)^3 = 8c^3$. $\therefore$ $8 | a^2$.

(what if c is odd? we'll see this can't happen.)

Well, if $a = 2d$ we'd have

$4d^2 = 8c^3$ or $d^2 = 2c^3$. $\therefore$ d is even.

So $d = 2f$. $\therefore$ $a = 2d = 2(2f) = 4f$.

$\therefore$ $4 \mid a$. $\therefore$ $16 \mid a^2$ $\therefore$ $16 \mid 8c^3$.

$\therefore$ $2 \mid c^3$. $\therefore$ c must be even.

$\therefore$ $c = 2g$. So, $b = 2c = 2(2g) = 4g$.

$\therefore$ $4 \mid b$.

(b) S $4 \mid b$. Then $b = 4g$.

$\therefore$ $a^2 = 64g^3$ $\therefore$ $64 \mid a^2$.

$\therefore$ $a^2 = 8^2 h$. $\therefore$ $2 \mid a$. $\therefore$ $a = 2r$

$\therefore$ $4r^2 = 8^2 h$. $\therefore$ $r^2 = 16 \cdot h$. $\therefore$ $2 \mid r$.

$\therefore$ $r = 2s$. $\therefore$ $4s^2 = 16h$. $\therefore$ $s^2 = 4h$.

$\therefore$ $2 \mid s$. $\therefore$ $s = 2t$. $\therefore$ $a = 2r =$

$= 2(2s) = 2(2(2t)) = 8t$. $\therefore$ $8 \mid a$.

(c) $a = 8$, $b = 4$

5. If $0<a$ is irrational, and $\frac{1}{a}$ is rational, then $\frac{1}{a} = \frac{k}{l}$ so $a = \frac{l}{k}$, which is a ratio of the integers l & k. So, a is rational, which is a contradiction. $\therefore$ $\frac{1}{a}$ is irrational.

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6. S $15^{\frac{1}{5}}$ is rational.

$\therefore$ $15^{\frac{1}{5}} = \frac{a}{b}$, where a, b have no common factors.

Then, $15b^5 = a^5$, $\therefore$ $3 \mid a^5$, so $3 \mid a$.

$\therefore$ $a = 3k$, $\therefore$ $15b^5 = (3k)^5 = 3 \cdot 3^4 \cdot k^5$.

$\therefore$ $5b^5 = 3^4 \cdot k^5$. $\therefore$ $3 \mid 5 \cdot b^5$.

$\therefore$ by problem 1, $3 \mid 5$ or $3 \mid b$.

$\therefore$ $3 \mid b$. $\therefore$ $3 \mid$ both a & b, a contradiction. Therefore $15^{\frac{1}{5}}$ is irrat.

7. (a) Prove by the 2^{nd} principal of (mathematical) induction.

we check $a_1 \le 5$, $a_2 \le 5^2$, $a_3 \le 5^3$.

$\mathcal{S}$ $a_k \le 5^k$ for all $k \le n$, (some $n \ge 3$).

Then

$$a_{n+1} = 2a_{n} + 3a_{n-1} \overset{SIH}{\le} 2 \cdot 5^n + 3 \cdot 5^{n-1}$$

$$= (10+3)\, 5^{n-1} \le 25 \cdot 5^{n-1} = 5^{n+1}.$$

$\therefore$ $a_n \le 5^n$ $\forall n \ge 1$.

(b) Do the same proof ~~with the~~ ~~for~~ $a_n \le 3^n$. $n = 1, 2, 3$ work.

$\mathcal{S}$ true for $k \le n$ (some $n \ge 3$)

Then $a_{n+1} = 2a_n + 3a_{n-1} \le 2 \cdot 3^n + 3 \cdot 3^{n-1}$

$$= (6+3)\, 3^{n-1} = 9 \cdot 3^{n-1} = 3^{n+1}.$$

$\therefore$ by the second princ. of math. ind. $a_n \le 3^n$ $\forall n \ge 1$.