(a) $p \rightarrow (q \lor r)$ is false iff
$$p \lor \neg q \lor \neg r$$
is false iff
$$p \land \neg q \land \neg r$$
is true.

Hence, the two statements are false (and, hence, true) at the same time.

(b) Using part (a) we know that
$$\neg (p \land b) \rightarrow [p \land a \lor p \land b]$$
is equivalent to
$$(p \land b \lor p \land a) \rightarrow p \land b.$$
would have to divide both b and a. But the only divisors of p are 1 and p. So, if we want a non-trivial common divisor, then since \( d = p \), since a \( \neq 0 \), recall we're assuming \( a \neq 0 \).

We can now apply the theorem given on the test. So we get \( x, y \in \mathbb{Z} \), s.t., \( ax + py = 1 \). Multiply both sides of the equation by b, we get:

\[
abx + pby = b.
\]

The left side is divisible by p, since \( p \mid ab\) and \( p \mid (pby) \). Hence, p must divide b. This is what we wanted.

(b) Proof by induction. We know the result for \( n=1 \) and \( n=2 \). Assume true for \( n \geq 2 \) and show for \( n+1 \). So we're assuming

\[
p \mid b, b_2, \ldots, b_n, b_{n+1}
\]

So, p divides a product of two natural numbers (\( b, b_n \) and \( b_{n+1} \)). By the \( n=2 \) case (proved in (b)).
we know that either
\[ p \mid (b_1 \ldots b_n) \quad \text{or} \quad p \mid b_{n+1} \]

but by the inductive hypothesis, if \( p \mid b_1 \ldots b_n \), then \( p \mid b \), or \( p \mid b_2 \), or \( \ldots \), or \( p \mid b_n \). So we have \( p \mid b \), or \( \ldots \), or \( p \mid b_n \) or \( p \mid b_{n+1} \).

2. (a) \[ \emptyset \in \mathcal{P}(A) = \mathcal{P}(B) \]. Then
\[ A \subseteq A \Rightarrow A \subseteq \mathcal{P}(A) = \mathcal{P}(B) \Rightarrow A \subseteq B. \]
Similarly, \( B \subseteq \mathcal{P}(B) = \mathcal{P}(A) \Rightarrow B \subseteq A \).
\[ \therefore A = B. \]

(b) \[ \mathcal{P}(\mathcal{P}(A)) = \mathcal{P}(\mathcal{P}(B)) \Rightarrow \mathcal{P}(A) = \mathcal{P}(B) \]
(by applying part (a) to the sets \( \mathcal{P}(A) \) and \( \mathcal{P}(B) \) in place of \( A \in B \)).
Now, again by part (a), since we now have \( P(A) = P(B) \), we get \( A = B \). □

3. \( P(A \cup B) \cap C = B^c \). If \( x \in A \cap B \cap C \), then \( x \in A \cup B \) (and \( x \in B \)) and \( x \in C \).
   \[ \therefore x \in (A \cup B) \cap C \subseteq B^c \quad \text{and} \quad \neq B \]
   So, \( x \in B \cap B = \emptyset \), a contradiction.
   \[ \therefore \text{there is no } x \in A \cap B \cap C. \]
   \[ \therefore A \cap B \cap C = \emptyset. \]

4. \( 2 | a = \Rightarrow 4 | a^2 \Rightarrow 4 | b^2 \Rightarrow 2 | b \).
   \[ \therefore b = 2c \text{ for some } c \in \mathbb{N}. \]
   \[ \therefore a^2 = b^3 = (2c)^3 = 8c^3 \Rightarrow 8 | a^2. \]
   (What if \( c \) is odd? We'll see this can't happen.)
   Well, if \( a = 2d \) we'd have...
\[ 4d^2 = 8c^3 \quad \text{or} \quad d^2 = 2c^3 \quad \therefore d \text{ is even} \]

So \( d = 2f \), \( a = 2d = 2(2f) = 4f \).

\[ 4 | a \quad \therefore 16 | a^3 \quad \therefore 16 | 8c^3 \]

\[ 2 | c^3 \quad \therefore c \text{ must be even} \]

\[ c = 2g \quad \therefore b = 2c = 2(2g) = 4g \]

\( 4 | b \).

(b) \( 8 | 4b \). Thus \( b = 4g \).

\[ a^2 = 64g^3 \quad \therefore 64 | a^2 \]

\[ a = 8g^2 \quad \therefore a^2 | a \quad \therefore a = 2r \]

\[ 4r^2 = 8g^2 \quad \therefore r^2 = 2g^2 \]

\[ 2 | r \quad \therefore s = 2 \quad \therefore a = 2r = 2(2s) = 2(2(2t)) = 8t \quad \therefore 8 | a \]

(c) \( a = 8 \quad b = 4 \)
So if \( \alpha \) is irrational, and \( \frac{1}{\alpha} \) is rational, then \( \frac{1}{\alpha} = \frac{5}{k} \) so \( \alpha = \frac{k}{5} \) which is a ratio of the integers \( k \) and \( 5 \).

So \( \alpha \) is irrational, which is a contradiction. Therefore \( \frac{1}{\alpha} \) is irrational.

b. \( 15^{\frac{1}{5}} \) is rational.

\[ 15^{\frac{1}{5}} = \frac{a}{b}, \text{ where } a, b \text{ have no common factors.} \]

Then \( 15b^5 = a^5 \), \( \Rightarrow 3 \mid a^5 \), \( \Rightarrow 3 \mid a \).

\[ a = 3k, \Rightarrow 15b^5 = (3k)^5 = 3^5 \cdot k^5. \]

\[ 5b^5 = 3^4 \cdot k^5, \Rightarrow 3 \mid 5b^5. \]

by problem 1, \( 3 \mid 5 \) or \( 3 \mid b \)

\[ 3 \mid b. \Rightarrow 3 \mid \text{ both } a = 3b, \]

a contradiction. Therefore \( 15^{\frac{1}{5}} \) is irrational.
7. We prove by the $2^{nd}$ principal of mathematical induction.

We check $a_1 \leq 5$, $a_2 \leq 5^2$, $a_3 \leq 5^3$.

By $a_k \leq 5^k$ for all $k \leq n$, \((n \geq 3)\).

Then

\[
a_{n+1} = 2a_n + 3a_{n-1} \leq 2 \cdot 5^n + 3 \cdot 5^{n-1}
\]

\[
= (10 + 3) 5^{n-1} \leq 25 \cdot 5^{n-1} = 5^{n+1}
\]

\[
\therefore \ a_n \leq 5^n \ \forall n \geq 1.
\]

(b) Do the same proof.

\[
a_n \leq 3^n. \ \ \ \forall n = 1, 2, 3 \ \ \text{work.}
\]

\[
\exists \ \text{true for } k \leq h \ (\text{same } n \geq 3)
\]

Then

\[
a_{n+1} = 2a_n + 3a_{n-1} \leq 2 \cdot 3^n + 3 \cdot 3^{n-1}
\]

\[
= (6 + 3) 3^{n-1} = 9 \cdot 3^{n-1} = 3^n
\]

\[
\therefore \ \text{by the second prin. of math. ind.} \ a_n \leq 3^n \ \forall n \geq 1.
\]