$\boxed{\text{I use } \mathcal{S} \text{ for "suppose"}}$

$f : X \to Y$

$\mathcal{P}(Y)$ is power set of Y

$\mathcal{P}(X)$ " " " " X,

4. $F : \mathcal{P}(Y) \to \mathcal{P}(X)$

$F(B) = f^{-1}(B)$

1. f 1-1 $\Rightarrow$ F onto

Pf Claim: f 1-1 $\Rightarrow$ $f^{-1}(f(A)) = A$.

Pf of claim. $\mathcal{S}$ $x \in f^{-1}(f(A))$. $\therefore f(x) \in f(A)$

$\therefore f(x) = f(a)$ for some $a \in A$.

$\therefore$ since f is 1-1, $x = a$. $\therefore x \in A$.

$\mathcal{S}$ $x \in A$. $\therefore f(x) \in f(A)$. $\therefore x \in f^{-1}(f(A))$

by definition of f^{-1}.

To show F is onto just note that for every A, $f^{-1}(f(A)) = A$. So taking $B = f(A)$, we have $F(B) = f^{-1}(B) = f^{-1}(f(A)) = A$. So any $A \in \mathcal{P}(X)$ is "hit" & $\therefore F$ is onto.

2. F onto $\Rightarrow$ f 1-1.

Pf $\mathcal{S}$ f is not 1-1. $\therefore \exists\, x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$

Let $A = \{x_1\}$. Since F is onto $\exists$ $B \in \mathcal{P}(Y)$, $F(B) = A$. That is, $f^{-1}(B) = A = \{x_1\}$.

$\therefore$ by definition of f^{-1}, and since $x_1 \in f^{-1}(B)$ (since $f^{-1}(B) = \{x_1\}$),

&

$f(x_1) \in B$. But $f(x_2) = f(x_1) \in B$. $\therefore f(x_2) \in B$

$\therefore x_2 \in f^{-1}(B) = \{x_1\}$. $\therefore x_2 = x_1$ $\rightarrow\leftarrow$

3. F 1-1 $\Rightarrow$ f onto

Pf $\mathcal{S}$ f is not onto. Let $B = f(X) = \{f(x) : x \in X\}$. Note that f not onto implies $B \neq Y$. But, $F(B) = f^{-1}(B)$. And, $u \in f^{-1}(B)$ iff $f(u) \in B$ iff $f(u) \in \{f(x) : x \in X\}$, which is always true ($\forall u \in X$).

$\therefore$ $f^{-1}(B) = X$. $\therefore$ $F(B) = X$. Also it is easy to see that $F(Y) = X$. But F is 1-1 so we would have $B = Y$. $\rightarrow\leftarrow$

4. f onto $\Rightarrow$ F 1-1.

Pf $\mathcal{S}$ $F(B_1) = F(B_2)$. We have to show $B_1 = B_2$. Suppose not, i.e., $B_1 \neq B_2$ $\therefore$ either $\exists \tilde{y} \in B_1 \setminus B_2$ or $\exists \tilde{y} \in B_2 \setminus B_1$. Since the argument is the same in both cases, let's assume $\tilde{y} \in B_1 \setminus B_2$. Since f is onto, $\exists$ $\tilde{x} \in X$, $f(\tilde{x}) = \tilde{y}$.

$\therefore$ $f(\tilde{x}) \in B_1 \setminus B_2$. But, $f(\tilde{x}) \in B_1$ is equivalent to $\tilde{x} \in f^{-1}(B_1) = F(B_1) = F(B_2) = f^{-1}(B_2)$. So, $\tilde{x} \in f^{-1}(B_2)$, which is equivalent to $\tilde{y} = f(\tilde{x}) \in B_2$. Contradicting $\tilde{y} \in B_1 \setminus \underline{B_2}$.

$\therefore$ $B_1 = B_2$. $\therefore$ F is 1-1