

1. Let $A \subseteq \mathbb{R}$. Define a function, $g(x) := \inf\{|x - y| : y \in A\}$.
 - (a) Prove that $g(x) = 0$ iff x is a limit point of a sequence from A .
 - (b) Prove that $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous.
2. If $f : (X, d) \rightarrow (Y, e)$ is continuous, then, for every closed set, $F \subseteq Y$, $f^{-1}(F)$ is closed.
3. Give an example of a metric space, (X, d) , and sets, $\{E_n\}$, with the finite intersection property, and, yet, $\bigcap_{n=1}^{\infty} E_n = \emptyset$.
4. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Assume also that for every $x \in [a, b]$, $f(x) > 0$. Show $\inf\{f(x) : x \in [a, b]\} > 0$.
5. Let $f(x) = x^4 + 2x^3 + 5x^2 - 6x - 10$. Show there exist a point, x , for which $f(x) = 0$.
6. If (X, d) is a metric space, F is a closed subset and $E \subseteq F$. Prove that $\bar{E} \subseteq F$, where \bar{E} denotes the closure of E .
7. Let (X, d) be a metric space, K a compact subset and U an open subset. If $K \subseteq U$, then there exists a $\delta > 0$ such that for every point, y for which there is an $x \in K$ with $d(x, y) < \delta$ (that is, those points in X which are within δ of some point in K), we have $y \in U$.
8. Let $f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$
Prove that f is continuous at $x = 0$.
9. Let $\{x_n\}$ be a sequence such that $x_1 \leq x_2$ and x_{n+1} is between x_n and x_{n-1} .
 - (a) Prove that $\{x_{2n+1}\}$ is an non-decreasing sequence and (similarly) $\{x_{2n}\}$ is an non-increasing sequence.
 - (b) Prove that the sequence $\{x_{2n+1}\}$ converges.
10. If $\{a_n^3 - \frac{1}{a_n^3}\}$ is bounded, then $\{a_n\}$ is bounded.
11. Prove that $\lim_{x \rightarrow 1} x^4 = 1$ using and ϵ, δ proof.
12. Using and ϵ, N proof, show that $\lim_{n \rightarrow \infty} \frac{n^3 - n + 10}{n^3 - 5n + 17} = 1$.
13. State and prove the Heine-Borel theorem.
14. Let $\{a_n\}$ be a sequence such that $|a_n - a_m| \geq 1$ if $m \neq n$. Prove that no subsequence converges. Why does this show that the sequence is unbounded?
15. State the intermediate value theorem. If $f(x)$ and $g(x)$ are continuous on $[0, 1]$ and

$$f(0) > g(0), \quad f(1) < g(1),$$

prove that there is a point $c \in (0, 1)$ where $f(c) - g(c) = 0$.

16. Show that for any increasing sequence of real numbers, the limit is the sup.
17. Let $b_n = \frac{1}{\ln(1+n)}$. Show that the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.
18. Let $a_1 = 2$ and for $n \geq 1$, $a_{n+1} = a_n + 6$. Find the limit and show that the sequence converges to that limit.
19. If $s_n = \sum_{j=1}^n \frac{1}{j^2}$, show that $\{s_n\}$ is a Cauchy sequence.
20. If $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are both continuous, prove that
- (a) $f + g$ is continuous.
 - (b) $f \cdot g$ is continuous.
 - (c) $f \circ g$ is continuous, where the circle denotes composition.
 - (d) f/g is continuous if $g(x) > 0$ for every $x \in \mathbb{R}$.