1. Let \( A \subseteq \mathbb{R} \). Define a function, \( g(x) := \inf \{|x - y| : y \in A\} \).

(a) Prove that \( g(x) = 0 \) iff \( x \) is a limit point of a sequence from \( A \).
(b) Prove that \( g : \mathbb{R} \rightarrow \mathbb{R} \) is continuous.

2. If \( f : (X, d) \rightarrow (Y, e) \) is continuous, then, for every closed set, \( F \subseteq Y, f^{-1}(F) \) is closed.

3. Give an example of a metric space, \((X, d)\), and sets, \( \{E_n\} \), with the finite intersection property, and, yet, \( \bigcap_{n=1}^{\infty} E_n = \emptyset \).

4. Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous. Assume also that for every \( x \in [a, b] \), \( f(x) > 0 \). Show \( \inf \{f(x) : x \in [a, b]\} > 0 \).

5. Let \( f(x) = x^4 + 2x^3 + 5x^2 - 6x - 10 \). Show there exist a point, \( x \), for which \( f(x) = 0 \).

6. If \((X, d)\) is a metric space, \( F \) is a closed subset and \( E \subseteq F \). Prove that \( \bar{E} \subseteq F \), where \( \bar{E} \) denotes the closure of \( E \).

7. Let \((X, d)\) be a metric space, \( K \) a compact subset and \( U \) an open subset. If \( K \subseteq U \), then there exists a \( \delta > 0 \) such that for every point, \( y \) for which there is an \( x \in K \) with \( d(x, y) < \delta \) (that is, those points in \( X \) which are within \( \delta \) of some point in \( K \)), we have \( y \in U \).

8. Let \( f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \)

   Prove that \( f \) is continuous at \( x = 0 \).

9. Let \( \{x_n\} \) be a sequence such that \( x_1 \leq x_2 \) and \( x_{n+1} \) is between \( x_n \) and \( x_{n-1} \).

   (a) Prove that \( \{x_{2n+1}\} \) is an non-decreasing sequence and (similarly) \( \{x_{2n}\} \) is an non-increasing sequence.
   (b) Prove that the sequence \( \{x_{2n+1}\} \) converges.

10. If \( \{a^3_n - \frac{1}{a^3_n}\} \) is bounded, then \( \{a_n\} \) is bounded.

11. Prove that \( \lim_{x \to 1} x^4 = 1 \) using and \( \epsilon, \delta \) proof.

12. Using and \( \epsilon, N \) proof, show that \( \lim_{n \to \infty} \frac{n^3 - n + 10}{n^3 - 5n + 17} = 1. \)


14. Let \( \{a_n\} \) be a sequence such that \( |a_n - a_m| \geq 1 \) if \( m \neq n \). Prove that no subsequence converges. Why does this show that the sequence is unbounded?

15. State the intermediate value theorem. If \( f(x) \) and \( g(x) \) are continuous on \([0, 1]\) and

   \( f(0) > g(0), \ f(1) < g(1), \)

   prove that there is a point \( c \in (0, 1) \) where \( f(c) - g(c) = 0. \)
16. Show that for any increasing sequence of real numbers, the limit is the sup.

17. Let \( b_n = \frac{1}{\ln(1 + n)} \). Show that the series \( \sum_{n=1}^{\infty} (-1)^n b_n \) converges.

18. Let \( a_1 = 2 \) and for \( n \geq 1 \), \( a_{n+1} = a_n + 6 \). Find the limit and show that the sequence converges to that limit.

19. If \( s_n = \sum_{j=1}^{n} \frac{1}{j^2} \), show that \( \{s_n\} \) is a Cauchy sequence.

20. If \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are both continuous, prove that
   
   (a) \( f + g \) is continuous.
   
   (b) \( f \cdot g \) is continuous.
   
   (c) \( f \circ g \) is continuous, where the circle denotes composition.
   
   (d) \( f/g \) is continuous if \( g(x) > 0 \) for every \( x \in \mathbb{R} \).