On the volume of the intersection of two L_p^n balls

by

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1. Introduction

This note deals with the following problem, the case $p=1$, $q=2$ of which was introduced to us by Vitali Milman: What is the volume left in the L_p^n ball after removing a t-multiple of the L_q^n ball? Recall that the L_r^n ball is the set $\{(t_1,t_2,\dots,t_n);\ t_i\in\mathbf{R},\ n^{-1}\sum_{i=1}^n|t_i|^r\le 1\}$ and note that for $0<p<q<\infty$ the L_q^n ball is contained in the L_p^n ball.

In Corollary 4 below we show that, after normalizing Lebesgue measure so that the volume of the L_p^n ball is one, the answer to the problem above is of order $e^{-ct^pn^{p/q}}$ for $T<t<\frac{1}{2}n^{\frac{1}{p}-\frac{1}{q}}$, where c and T depend on p and q but not on n.

The main theorem, Theorem 3, deals with the corresponding question for the surface measure of the L_p^n sphere. Theorem 3 and Corollary 4 together with some other remarks form Section 3. In Section 2 we introduce a class of random variables to be used in the proof of the main theorem. These random variables are related to L_p in the same way that Gaussian variables are related to L_2.

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2. Preliminaries

Here we introduce a class of random variables to be used in the proof of the main theorem and summarize some of their properties. Fix a $0 < p < \infty$ and let $x, x_1, x_2, \ldots, x_n$ be independent random variables each with density function $c_p e^{-t^p}$, $t > 0$. Note that necessarily $c_p = p/, (1/p)$. The first claim is known, though we could not locate a reference.

Lemma 1. *Put* $S = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$, *then* $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ *is uniformly distributed over the positive quadrant of the sphere of* l_p^n, *i.e., over the set* $\Delta_p = \{(t_1, t_2, \ldots, t_n)\,;\ t_i \geq 0,\ \sum t_i^p = 1\}$ *equipped with the* $(n-1)$*-dimensional normalized Lebesgue measure. Moreover,* $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ *is independent of* S.

Proof. For any Borel subset A of Δ_p,

$$
\begin{aligned}
&P\Big(\Big(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\Big) \in A \Big| S = a\Big) = \\
&\quad = \lim_{\epsilon \to 0} \frac{P((x_1, \ldots, x_n) \in \mathbf{R}_+ A \ \&\ a - \epsilon \leq S \leq a + \epsilon)}{P(a - \epsilon \leq S \leq a + \epsilon)} \\
&\quad = \lim_{\epsilon \to 0} \int_{\substack{(t_1, \ldots, t_n) \in \mathbf{R}_+ A \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} e^{-\sum t_i^p}\, dt \Big/ \int_{\substack{(t_1, \ldots, t_n) \in \mathbf{R}_+^n \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} e^{-\sum t_i^p}\, dt \\
&\quad \leq \limsup_{\epsilon \to 0} e^{-(a-\epsilon)^p + (a-\epsilon)^p\big)^p} \int_{\substack{(t_1, \ldots, t_n) \in \mathbf{R}_+ A \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} dt \Big/ \int_{\substack{(t_1, \ldots, t_n) \in \mathbf{R}_+^n \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} dt \\
&\quad = \lambda(A),
\end{aligned}
$$

where λ is the normalized Lebesgue measure on Δ. Similarly,

$$
P\Big(\Big(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\Big) \in A \Big| S = a\Big) \geq \lambda(A).
$$

This proves that $P\Big(\Big(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\Big) \in A\Big) = \lambda(A)$ and that $\Big(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\Big)$ is independent of S. ■

In the next claim we gather some more properties of the random variables x_i.

Lemma 2. *Let $x, x_1, \dots, x_n$ be as above, then*

1. *c_p is bounded away from zero and infinity when $p \to \infty$.*
2. *For all $h > 0$ and all $0 < p < \infty$, $\mathbf{E}e^{-hx^p} = \left(\frac{1}{1+h}\right)^{1/p}$. In particular,*

$$\mathbf{E}e^{-hx^p} \geq e^{-h/p} \quad \textit{for all} \quad h > 0 \quad \textit{and} \quad \mathbf{E}e^{-hx^p} \leq e^{-h/2p} \quad \textit{for all} \quad 0 < h \leq 1.$$

3. *For all $0 < u < \infty$ and all $0 < p < \infty$, $P(x^p > u) \geq \frac{c_p}{2p} e^{-2u}$. If $p \geq 1$ and $u \geq 1$, then also $P(x^p > u) \leq \frac{c_p}{p} e^{-u/2}$. In particular, for $p \geq 1$ and all u, $P(x^p > u) \leq Ce^{-u/2}$ for some universal C.*
4. *For all $1 \leq p \leq q < \infty$, $\mathbf{E}\left(\sum_{i=1}^n x_i^q\right)^{1/q}$ is equivalent, with universal constants, to*

$q^{1/p} n^{1/q}$, if $q \leq \log n$, and to $(\log n)^{1/p}$ otherwise.

Proof. 1. Follows easily from the fact that $c_p = p/, (1/p) = , (\frac{1}{p} + 1)^{-1}$.

2. is a simple computation.

3. is also simple, here is a sketch of the proof.

$$\begin{aligned} P(x^p > u) &= c_p \int_{u^{1/p}}^{\infty} e^{-t^p} dt \\ &\geq c_p \int_{u^{1/p}}^{(u+1)^{1/p}} \frac{pt^{(p-1)}}{p(u+1)^{(p-1)/p}} e^{-t^p} dt \\ &= \frac{c_p}{p(u+1)^{(p-1)/p}} \left(1 - \frac{1}{e}\right) e^{-u} \\ &\geq \frac{c_p}{2p(u+1)} e^{-u} \\ &\geq \frac{c_p}{2p} e^{-2u}. \end{aligned}$$

The other inequality in 3 is proved in a similar way.

4. First note that for all $0 < p, q < \infty$

$$\mathbf{E}x^q = c_p \int_0^{\infty} t^q e^{-t^p} dt = \frac{c_p}{p}, \left(\frac{q+1}{p}\right)$$

so that, by the triangle inequality and 1, if $1 \le p \le q < \infty$

$$\mathbf{E}\Big(\sum_{i=1}^{n} x_i^q\Big)^{1/q} \le \Big(\sum_{i=1}^{n} \mathbf{E}x_i^q\Big)^{1/q} = \Big(\frac{c_p}{p}, \Big(\frac{q+1}{p}\Big)\Big)^{1/q} n^{1/q} \le Cq^{1/p}n^{1/q}$$

for some universal C. For the lower bound in the case $q \le \log n$, divide $\{1,2,\dots,n\}$ into approximately n/e^q disjoint sets of cardinality approximately e^q each, then

$$\begin{aligned}
\mathbf{E}\Big(\sum_{i=1}^{n} x_i^q\Big)^{1/q} &= \mathbf{E}\Big(\sum_{j}\Big(\sum_{i\in\sigma_j} x_i^q\Big)^{q/q}\Big)^{1/q} \\
&\ge \mathbf{E}\Big(\sum_{j}\big(\max_{i\in\sigma_j} x_i\big)^q\Big)^{1/q} \\
&\ge \Big(\sum_{j}\big(\mathbf{E}\max_{i\in\sigma_j} x_i\big)^q\Big)^{1/q} \\
&\ge c'(\log e^q)^{1/p}(n/e^q)^{1/q} \\
&\ge c'' q^{1/p} n^{1/q}.
\end{aligned}$$

Now, for the case $q > \log n$ we note first that, by 3,

$$P(\max_{1\le i\le n} x_i > t) \ge 1 - \big(1 - \frac{c_p}{2p}e^{-2t^p}\big)^n.$$

For n smaller than an absolute multiple of p, the lower bound follows easily from the fact that $\mathbf{E}x_1$ is larger that a universal positive constant, so assume that $n \ge 20p/c_p$ and put $t = 2^{-1/p}\Big(\log\frac{nc_p}{2p}\Big)^{1/p}$.Then, for some universal c,

$$P(\max_{1\le i\le n} x_i > c(\log n)^{1/p}) \ge 1/2.$$

In particular, $\mathbf{E}\max_{1\le i\le n} x_i \ge c(\log n)^{1/p}$, which implies the lower bound in this case since $\Big(\sum_{i=1}^n x_i^q\Big)^{1/q}$ is universally equivalent to $\max_{1\le i\le n} x_i$. The upper bound in this case, though a bit harder, is also standard and since we don't use it in the sequel we shall leave it to the reader. ∎

The statement in 4, for the case $p = 2$, was noticed by the first named author several years ago while seeking a precise estimate for the dimension of the Euclidean sections of l_p^n spaces (see [MS] p.145 Remark 5.7). The original proof was more complicated. The proof presented here is an adaptation of a proof of the case $p = 2$ shown to us by J. Bourgain.

3. The main result

Theorem 3. *For all $1 \leq p < q < \infty$ there are constants $c = c(p,q)$, $C = C(p,q)$ and $T = T(p,q)$ such that if μ denotes the normalized Lebesgue measure on the positive quadrant of the unit sphere of L_p^n then*

$$\mu(\|u\|_{L_q^n} > t) \leq \exp(-ct^p n^{p/q}) \tag{1}$$

for all $t > T$, and

$$\mu(\|u\|_{L_q^n} > t) \geq \exp(-Ct^p n^{p/q}) \tag{2}$$

for all $2 \leq t \leq \frac{1}{2} n^{\frac{1}{p}-\frac{1}{q}}$.
Moreover, for $q > 2p$ (or any other universal positive multiple of p), one can take $c(p,q) = \frac{\gamma}{p}$, $C(p,q) = \frac{\Gamma}{p}$ and $T(p,q) = \tau \min\{q, logn\}^{1/p} \leq q^{1/p}$. Here γ ,, and τ are universal constants.

Proof. By Lemma 1 above,

$$\mu(\|u\|_{L_q^n} > t) = P(n^{\frac{1}{p}-\frac{1}{q}}(\sum_{i=1}^n x_i^q)^{1/q}/(\sum_{i=1}^n x_i^p)^{1/p} > t)$$

where x_i are independent random variables each with density $c_p e^{-t^p}$. Assume, for the simplicity of the presentation, that n is even. Put $S = (\sum_{i=1}^n x_i^p)^{1/p}$ and let p_j, $j = 1, 2, \ldots, n/2$ be positive numbers with sum $\leq 1/2$. Then

$$P\big(n^{\frac{1}{p}-\frac{1}{q}}(\sum_{i=1}^n x_i^q)^{1/q}/(\sum_{i=1}^n x_i^p)^{1/p} > t\big) =$$

$$= P\Big(\sum_{i=1}^{n} x_i^q > \frac{t^q(\sum_{i=1}^n x_i^p)^{q/p}}{n^{\frac{q}{p}-1}}\Big)$$

$$\leq \sum_{i=1}^{n/2} P(x_i^* > tp_i^{1/q} S/n^{\frac{1}{p}-\frac{1}{q}}) + P\Big(\sum_{i=\frac{n}{2}+1}^{n} x_i^{*q} > t^q S^q / 2n^{\frac{q}{p}-1}\Big) \qquad (3)$$

where $\{x_j^*\}$ denotes the nonincreasing rearrangement of $\{|x_j|\}$.

Since

$$\sum_{j=\frac{n}{2}+1}^{n} x_i^{*q} \leq \frac{n}{2} x_{\frac{n}{2}}^{*q} \leq \frac{n}{2}\Big(\frac{2}{n}\sum_{i=1}^{n/2} x_i^{*p}\Big)^{q/p}$$

$$\leq 2^{\frac{q}{p}-1} S^q / n^{\frac{q}{p}-1} \quad ,$$

we get that, if $t \geq 2^{1/p}$, the second term in (3) is zero.

To evaluate the first term in (3), fix $1 \leq j \leq n/2$. Then,

$$P(x_j^* > tp_j^{1/q} S/n^{\frac{1}{p}-\frac{1}{q}}) \leq \binom{n}{j} P(x_1, \ldots, x_j > tp_j^{1/q} S/n^{\frac{1}{p}-\frac{1}{q}})$$

$$\leq \binom{n}{j} P\big(x_1^p, \ldots, x_j^p > t^p p_j^{p/q} \sum_{i=j+1}^{n} x_i^p / n^{1-p/q}\big).$$

From Lemma 2 (first 3 and then 2) we get that the last expression is dominated by

$$\binom{n}{j} C^j \mathbf{E} \exp\big(-j p_j^{p/q} t^p \sum_{i=j+1}^{n} x_i^p / n^{1-p/q}\big)$$

$$\leq \binom{n}{j} C^j \exp(-j p_j^{p/q} t^p (n-j)/2pn^{1-p/q})$$

for some universal C. Note that the last inequality holds if $jn^{p/q-1} p_j^{p/q} t^p \leq 1$. If this is not the case the probability we are trying to evaluate is zero. Finally, the last term is dominated by

$$\exp\Big(j\big(log\frac{en}{j} + C - \frac{p_j^{p/q} t^p n^{p/q}}{4p}\big)\Big). \qquad (4)$$

Now, for α to be chosen momentarily, let p_j, $j=1,\dots,n/2$, be such that

$$j(log\frac{en}{j}+C-\frac{p_j^{p/q}t^pn^{p/q}}{4p})=-\alpha n^{p/q}t^p$$

i.e.,

$$p_j=(4p\frac{log\frac{en}{j}}{t^pn^{p/q}}+\frac{4Cp}{t^pn^{p/q}}+\alpha\frac{4p}{j})^{q/p}.$$

We thus get that,for some universal constant C,

$$p_j\le 2^{\frac{q}{p}-1}\frac{(Cp)^{q/p}(log\frac{en}{j})^{q/p}}{t^qn}+2^{\frac{q}{p}-1}\alpha^{q/p}\frac{(4p)^{q/p}}{j^{q/p}}. \tag{5}$$

It is easy to see that, for $1\le p<q<\infty$,

$$\sum_{j=1}^{n/2}(log\frac{en}{j})^{q/p}\le An\min\{q^{q/p},(logn)^{q/p}\}$$

for some universal A. Thus the sum over j of the first terms in (5) is smaller than $1/4$ if, for some universal γ, $t>\gamma\min\{q^{1/p},(\log n)^{1/p}\}$. The sum over j of the second terms in (5) is bounded by $1/4$ if $\alpha<B\frac{1}{p}(\frac{q}{p}-1)^{p/q}$, for some universal B. Choosing α to satisfy this inequality and using (3),(4) and (5) we get that, for $t>\gamma\min\{q^{1/p},(logn)^{1/p}\}$,

$$\mu(\|u\|_{L_q^n}>t)\le\frac{n}{2}e^{-\alpha n^{p/q}t^p}.$$

Under the conditions on t, the factor $n/2$ can be absorbed in the second term (changing α to another constant of the same order of magnitude as a function of p), thus proving (1).

We now turn to the proof of the lower bound (2) which is simpler. Using Claim 1 again,

$$\begin{aligned}\mu\|u\|_{L_q^n}>t)&=P(n^{\frac{1}{p}-\frac{1}{q}}(\sum_{i=1}^n x_i^q)^{1/q}/(\sum_{i=1}^n x_i^p)^{1/p}>t)\\&\ge P(x_1>St/n^{\frac{1}{p}-\frac{1}{q}})\\&=P\Big(x_1>\frac{t}{(n^{(1-p/q)}-t^p)^{1/p}}(\sum_{i=2}^n x_i^p)^{1/p}\Big).\end{aligned}$$

Since $t^p \leq \frac{1}{2}n^{(1-p/q)}$, this dominates

$$P\Big(x_1 > \frac{2^{1/p}t}{n^{\frac{1}{p}-\frac{1}{q}}}(\sum_{i=2}^n x_i^p)^{1/p}\Big).$$

Now, by Claim 2.3.,

$$\begin{aligned} P\Big(x_1 > \frac{2^{1/p}t}{n^{\frac{1}{p}-\frac{1}{q}}}(\sum_{i=2}^n x_i^p)^{1/p}\Big) &\geq \frac{c_p}{2p}\mathbf{E}\exp(-4t^p\sum_{i=2}^n x_i^p/n^{(1-p/q)}) \\ &= \frac{c_p}{2p}(\mathbf{E}\exp(-4t^p x_1^p/n^{(1-p/q)}))^{n-1} \\ &= \frac{c_p}{2p}\Big(\frac{1}{1+\frac{4t^p}{n^{1-p/q}}}\Big)^{(n-1)/p} \qquad \textit{(by Claim 2.2.)} \\ &\geq \frac{c_p}{2p}\exp\Big(-\frac{4t^p(n-1)}{pn^{(1-p/q)}}\Big) \\ &\geq \frac{c_p}{2p}e^{4t^p n^{p/q}/p}. \end{aligned}$$

Finally observe that, since c_p is bounded away from zero and $t \geq 2$, the factor $\frac{c_p}{2p}$ can be absorbed in the second term (changing 4 to another universal constant). ■

Remarks:

1. It follows from the proof that, for n large enough and q close to p, one can take $c(p,q) = \frac{c}{p}\Big(\frac{q}{p} - 1\Big)$ for some universal constant c.

2. It follows from the statement of the theorem that, for $q = \infty$,

$$\mu(\|u\|_\infty > t) \leq e^{-\gamma t^p/p}$$

for all $t > \tau(\log n)^{1/p}$, and

$$\mu(\|u\|_\infty > t) \geq e^{-\Gamma t^p/p}$$

for all $2 \leq t \leq \frac{1}{2}n^{\frac{1}{p}}$, where γ, , and τ are universal constants.

3. Note that it follows from Claim 1 and Claim 2.4. that the order of magnitude of T is the correct one.

4. The restriction $p \geq 1$ in Theorem 3 above and in Corollary 4 below can be replaced by $p > 0$ if one replaces the inequality $t \geq 2$ with $t \geq d$, for some d depending only on p and q, and removes the "moreover" part. We didn't check the dependence of the constants on p and q in this case.

The last remark is that one can get a similar statement for the full balls. We state it as a corollary.

Corollary 4. *For all $1 \leq p < q < \infty$ there are constants $c = c(p,q)$, $C = C(p,q)$ and $T = T(p,q)$ such that if ν denotes the normalized Lebesgue measure on the ball of L_p^n then, for all n large enough,*

$$\nu(\|u\|_{L_q^n} > t) \leq \exp(-ct^p n^{p/q}) \tag{6}$$

for all $t > T$, and

$$\nu(\|u\|_{L_q^n} > t) \geq \exp(-Ct^p n^{p/q}) \tag{7}$$

for all $2 \leq t \leq \frac{1}{2} n^{\frac{1}{p}-\frac{1}{q}}$. Moreover, for $q > 2p$ (or any other universal positive multiple of p), one can take $c(p,q) = \frac{\gamma}{p}$, $C(p,q) = \frac{\Gamma}{p}$ and $T(p,q) = \tau \min\{q, logn\}^{1/p} \leq q^{1/p}$, where γ ,, and τ are universal constants.

The proof follows easily from Theorem 3 and the formula

$$\nu(A) = n \int_0^1 r^{n-1} \mu\Big(\frac{A}{r}\Big) dr$$

which holds for all Borel sets A in the ball of L_p^n.

References

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