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Central Limit Theorems and Weak Laws of Large Numbers in Certain Banach Spaces

Evarist Giné[1*] and Joel Zinn[2*,**]

[1] Louisiana State University, Department of Mathematics, Baton Rouge, LA 70803, U.S.A.
[2] Michigan State University, Department of Statistics and Probability,
East Lansing, MI 48824, U.S.A.

Summary. For B a type 2 Banach lattice, we obtain a relationship between the central limit theorem in B and the weak law of large numbers (for the sum of the squares of the random vectors) in another Banach lattice $B_{(2)}$. We then obtain some two-sided estimates for $E\|S_n\|^p$ which in l_p spaces, $1 \leqq p < \infty$, give n.a.s.c. for the weak law of large numbers. As a consequence of these estimates we also solve the domain of attraction problem in l_p, $p<2$. Several examples and counterexamples are provided.

1. Introduction

Pisier and Zinn ([29], Theorem 5.1) obtained necessary and sufficient conditions for l_p-valued random variables, $p>2$, to satisfy the central limit theorem (i.i.d., $n^{\frac{1}{2}}$ case, which we will often denote as CLT) and, as remarked at the end of [29], their theorem extends as well to a certain class of Banach lattices of type 2 (the p-convex, q-concave Banach lattices, $p>2$, $q<\infty$; definitions are given below). This result was generalized to infinitesimal systems of L_p-valued r.v.'s, $p>2$, in [11]. Here we continue working on this subject with the objective of finding interesting necessary and sufficient conditions for a B-valued r.v. X to satisfy the CLT.

In [29] it is asked whether the conditions X pregaussian and $nP\{\|X\|>n^{\frac{1}{2}}\}\to 0$ are necessary and sufficient for $X\in$CLT in type 2 spaces. It is proved in Sect. 4 below that these conditions imply the CLT in $l_2(B)$ if and only if B is of cotype 2, in particular this is not true in $l_2(l_p)$, $p>2$. The question is thus answered in the negative.

Then the problem arises of finding n.a.s.c. for $X\in$CLT in type 2 Banach spaces not covered by the CLT in [29]. We obtain two types of such conditions. The first one, in Sect. 2, is concerned with the generalization to Banach

* Partially supported by National Science Foundation Grants
** Presently at Texas A & M University, Department of Mathematics, College Station, TX 77843, U.S.A.

spaces of the obvious relation $X \in \mathrm{CLT} \Leftrightarrow X^2 \in \mathrm{WLLN}$ (weak law of large numbers) for real valued r.v.'s. We show that with the natural definition of $|X|^2$, this equivalence characterizes Banach lattices of type 2 (Theorem 2.9). We also include the relation between weak convergence of $\sum_j X_{nj}$ and $\sum_j |X_{nj}|^2$ for general infinitesimal arrays in type 2 Banach spaces with an unconditional basis.

The second set of conditions, in Sect. 5, is explicitly in terms of the distribution of X, but applies only to spaces of the form $l_2(l_p)$, $p>2$. These are interesting conditions because they are not "classical": they involve quantiles and truncated moments of the coordinates of X (Theorem 5.6). We use the CLT-WLLN relation to obtain them, but the main ingredient is a two-sided estimation of $E\left\|\sum_{i=1}^{\infty} X_i\right\|^p$, X_i independent, symmetric B-valued r.v.'s, given in Sect. 3 (Theorems 3.3, 3.4). We believe that this estimation has independent interest. Precedents for Theorem 3.3 are to be found in [17], which prove sharp two-sided bounds for $E\left|\sum_{i=1}^{n} X_i\right|$, X_i i.i.d. real valued, and in [20] which extend Klass's bounds to type and cotype 2 spaces (different sides of the inequalities for different types of spaces). The bounds here, in the i.i.d. real case, are equivalent up to constants to Klass's bounds (and to Kuelbs and Zinn's bounds in the type and cotype 2 cases), perhaps less precise, but simpler to evaluate. The proof of these bounds rely solely on an inequality of Hoffmann-Jørgensen [13].

The estimation given in Theorem 3.3 is applicable in other situations. To illustrate this point (as well as for the interest of the subject) we apply it to obtain the weak law of large numbers in l_1 (and in $l_1(l_p)$, in the process of obtaining the CLT in $l_2(l_p)$) and to solve the domains of attraction problem in l_p, $p<2$. So, in Sect. 6, we give n.a.s.c. for a l_p-valued r.v. X to be in the domain of attraction (DA) of a r-stable law, $r \geqq p$ (the case $r<p$ is well known), and apply the results to several examples. In fact, as we point out below, Theorem 3.3 can be used to obtain a general central limit theorem in l_p, $1 \leqq p<2$, much in the same way as Rosenthal's inequalities were used in [11] for the case $p>2$.

In Sect. 2 we review all we use on Banach lattices and include some examples (l_p, Orlicz and Lorentz spaces). The source is mainly [22].

Next we give some notation. All Banach spaces in this article are assumed to be separable, even without explicit mention. If X is a B-valued r.v., $\{X_i\}$ denotes a sequence of independent copies of X, and likewise for real valued r.v.'s η. $X \in \mathrm{CLT}$ means, as usual, that there is a Gaussian p.m. γ on B such that

$$\mathscr{L}\left(\sum_{i=1}^{n} X_i/n^{\frac{1}{2}}\right) \to_w \gamma$$

where $\to_w$ denotes weak convergence (as $n \to \infty$). $X \in \mathrm{WLLN}$ will mean that there exists $a \in B$ such that

$$\sum_{i=1}^{n} X_i/n \to_{pr} a$$

where $\to_{pr}$ denotes convergence in probability. $X \in \mathrm{DNA}_r$, $r<2$, will mean that there exists a r-stable law ρ on B such that for some constants $b_n \in B$,

$$\mathscr{L}\left(\sum_{i=1}^{n} X_i/n^{1/r} - b_n\right) \to_w \rho.$$

It is well known that for $r>1$ and X centered, and for $r<1$, b_n can be taken to be zero (by taking a shift of ρ if necessary). If the norming constants are a_n instead of $n^{1/r}$ we write $X \in \mathrm{DA}_r(a_n)$, $r \leqq 2$, meaning that X is in the domain of attraction of a r-stable law with norming $\{a_n\}$. The canonical basis of l_p is denoted by $\{e_\alpha\}$ and X_α denotes the α-th coordinate of X, i.e. $X = \sum_\alpha X_\alpha e_\alpha$. In that case, $\{X_{\alpha i}\}_{i=1}^{\infty}$ will denote independent copies of X_α. For $0<p<\infty$, we let $D_p = \{f \in (l_p)' \colon f = \sum_{\alpha \in I} a_\alpha e_\alpha^*,\ a_\alpha \in \mathbb{R},\ I \in \mathbb{N}$ a finite set, e_α^* the linear form conjugate to e_α, $\alpha = 1, \ldots\}$. D_p is sequentially w^*-dense in $(l_p)'$, and this makes this set useful in weak convergence. $\{\varepsilon_i\}$ will denote a Rademacher sequence, i.e. a sequence of i.i.d. r.v.'s such that $P\{\varepsilon_i = 1\} = P\{\varepsilon_i = -1\} = \frac{1}{2}$; and ε will stand for a Rademacher r.v. Rademacher sequences or r.v.'s will always be independent of the rest of the variables in the context where they will be used.

2. CLT and WLLN in 2-convex Banach Lattices

2i. Preliminaries on p-convex and q-concave Banach Lattices

For the reader's convenience we summarize some notation, definitions and propositions on convex and concave Banach lattices (general reference: [22]). A Banach lattice is called *p-convex*, $1 \leqq p < \infty$, if there exists a constant $M<\infty$ such that for any finite set $\{x_i\} \subset B$

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \right\| \leqq M \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}, \tag{2.1}$$

and it is called *p-concave*, $p \geqq 2$, if the reversed inequality holds. If B is a Banach lattice of real functions with the natural ordering ($f \geqq 0 \Leftrightarrow f(t) \geqq 0$ for all t) then the function

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}(t) = \left(\sum_{i=1}^{n} |x_i(t)|^p\right)^{1/p}$$
$$= \mathrm{l.u.b.}\left\{\sum_{i=1}^{n} a_i x_i(t) \colon \sum_{i=1}^{n} |a_i|^q \leqq 1,\ 1/p + 1/q = 1\right\}$$

is an element of B, and this generalizes in a suitable way to any Banach lattice: given $x_1, \ldots, x_n \in B$, there is a natural correspondence between $f(x_1, \ldots, x_n)$ and $f(t_1, \ldots, t_n)$ if $f \colon \mathbb{R}^n \to \mathbb{R}$ is continuous and defined only by linear and lattice operations; this correspondence extends by taking limits to continuous homogeneous functions (see e.g. [22], p. 40–43). It is in this way that (2.1) makes sense. There are several reasons why convex and concave Banach lattices are interesting to us. One reason is that in 2-convex Banach lattices there is a way of "squaring" vectors so that the squared vectors form another Banach lattice with norm $|||\cdot|||$ such that $|||x^2||| \simeq \|x\|^2$. Specifically, given a 2-convex Banach lattice B, the Banach lattice $B_{(2)}$ of its squares is defined by:

1) $B_{(2)} = B$ as ordered sets;

2) if ϕ is the identification map $\phi: B \to B_{(2)}$ then the linear operations are $\phi(x) + \phi(y) = \phi((x^2 + y^2)^{\frac{1}{2}})$, $\alpha\phi(x) = \phi(\alpha^{\frac{1}{2}} x)$ for all $x, y \in B$, $\alpha \in \mathbb{R}$;

3) $|||\phi(x)||| = \inf\{\sum \|x_i\|^2 : |\phi(x)| = \sum |\phi(x_i)|,\ x_i \in B$, finite sums$\}$.

Here $(x^2 + y^2)^{\frac{1}{2}}$ corresponds to the homogeneous function

$$|t_1^2 \operatorname{sgn} t_1 + t_2^2 \operatorname{sgn} t_2|^{\frac{1}{2}} \operatorname{sgn}(t_1^2 \operatorname{sgn} t_1 + t_2^2 \operatorname{sgn} t_2)$$

by the above mentioned functional calculus, and $\alpha^{\frac{1}{2}} = |\alpha|^{\frac{1}{2}} \operatorname{sgn} \alpha$. With these operations and norm $B_{(2)}$ is a Banach lattice and there exists $c \in (0, \infty)$ such that

$$c^{-1} |||\phi(x)||| \leqq \|x\|^2 \leqq c\, |||\phi(x)|||. \tag{2.2}$$

If the convexity constant is 1, then $|||\phi(x)||| = \|x\|^2$. The following notation will be in force:

$$\phi(x) = x^2, \qquad \phi(|x|) = |x|^2. \tag{2.3}$$

If B is a Banach space with an unconditional basis $\{e_i\}$ then it can be renormed in such a way that B is a Banach lattice with the ordering induced by the basis (namely, $\sum a_i e_i \geqq 0 \Leftrightarrow a_i \geqq 0$ for all i). In that case $B_{(2)}$ can be identified to the space of sequences $\{b_i\}$ such that $\sum b_i^2 e_i \in B$,

$$\phi(\textstyle\sum a_i e_i) = \sum a_i^2 (\operatorname{sgn} a_i)\, \phi(e_i) \quad \text{and} \quad |||\phi(\sum a_i e_i)||| \simeq \|\sum a_i e_i\|^2.$$

For instance, if $B = l_p$, $p \geqq 2$, then $B_{(2)} = l_{p/2}$. If the unconditionality constant of $\{e_i\}$ is 1 then no renorming is necessary ([22], p. 2). So, in this case, as well as in the case of spaces of functions, $B_{(2)}$ has a very natural meaning. The notion of $B_{(2)}$ is due to Krivine ([19]).

These spaces have some interesting properties. We list two very useful ones ([22], p. 49–50 and p. 46; originally due to Maurey [25]).

2.1. Theorem. *Let B be a Banach lattice and $\{\varepsilon_i\}$ a Rademacher sequence. Then there exists $C \in (0, \infty)$ such that for any finite collection $\{x_i\} \subset B$,*

$$C^{-1} \|(\textstyle\sum |x_i|^2)^{\frac{1}{2}}\| \leqq E \|\sum \varepsilon_i x_i\| \leqq C \|(\sum |x_i|^2)^{\frac{1}{2}}\|$$

if and only if B is q-concave for some $q < \infty$. The left side inequality holds in any Banach lattice.

An immediate corollary is that under the conditions of the theorem, if $\{X_i\}$ are independent, symmetric B-valued r.v.'s, then

$$C^{-1} E \|(\textstyle\sum |X_i|^2)^{\frac{1}{2}}\| \leqq E \|\sum X_i\| \leqq C E \|(\sum |X_i|^2)^{\frac{1}{2}}\|. \tag{2.4}$$

2.2. Theorem. *A Banach lattice B is of type 2 if and only if it is p-convex and q-concave for some $p \geqq 2$ and $q < \infty$.*

See e.g. [22], p. 100, for the complete set of relations between type, cotype, convexity, concavity and moduli of convexity and smoothness.

Next we consider some examples. $l_p(B)$ denotes the space of sequences $\{x_i\} \subset B$ such that $\sum \|x_i\|^p < \infty$, with norm $(\sum \|x_i\|^p)^{1/p}$. Since the l_p spaces are p-convex and p-concave we have:

2.3. Proposition. *The spaces $l_{p_1}(l_{p_2} \dots (l_{p_k}) \dots)$ are*

$$\min(p_1, \dots, p_k)\text{-convex} \quad \text{and} \quad \max(p_1, \dots, p_k)\text{-concave}.$$

An Orlicz function N satisfying the Δ_2-condition is a continuous, convex, non-decreasing function on $[0, \infty)$ such that $N(0)=0$, $N(t)>0$ if $t>0$, $\lim_{t\to\infty} N(t) = \infty$ and $\sup_{t>0} N(2t)/N(t) < \infty$. Given such a function, the Orlicz sequence space l_N is a separable Banach space defined as

$$l_N = \{\{x_i\} \in \mathbb{R}^{\mathbb{N}}: \textstyle\sum N(|x_i|) < \infty\},$$

with

$$\|\{x_i\}\|_N = \inf\{\rho > 0: \textstyle\sum N(|x_i|/\rho) \leqq 1\}.$$

2.4. Proposition. 1) *Let N be an Orlicz function satisfying the Δ_2-condition and let $M(t) = \int_0^t N(x) x^{-1} dx$. Then l_N is of type 2 if and only if*

$$\sup_{0<u,v\leqq 1} M(uv)/u^2 M(v) < \infty. \tag{2.5}$$

2) *If N is an Orlicz function satisfying the Δ_2-condition and such that $N(t^{\frac{1}{2}})$ is equivalent to a convex function $\tilde{N}$ (i.e. $N(c_1 t^{\frac{1}{2}}) \leqq \tilde{N}(t) \leqq N(c_2 t^{\frac{1}{2}})$ for some c_1, $c_2 \in (0, \infty)$ and all $t>0$), then l_N is of type 2 and $(l_N)_{(2)} \simeq l_{\tilde{N}}$. Under these conditions, l_N is p-convex and q-concave for some $p>2$ and $q<\infty$ if and only if there exists $\lambda > 1$ and $v_0 > 0$ such that for all $v < v_0$,*

$$\tilde{N}(v) \leqq (2\lambda)^{-1} \tilde{N}(\lambda v). \tag{2.6}$$

Proof (highly non-selfcontained!). By [21] 4.a.4 and 4.a.9, and [22] 2.b.5 and p. 100, l_N is q-concave as long as N satisfies the Δ_2-condition. In this case type 2 is equivalent to a modulus of smoothness of power type 2 ([22] p. 100), and therefore, part 1) follows from [8], Propositions 19 and 21 and Lemma 20. A simple computation using just the definition shows that if N satisfies the hypothesis in 2) then l_N is 2-convex (hence of type 2 because it is also q-concave) and $(l_N)_{(2)} \simeq l_{\tilde{N}}$. From [22], p. 53, 54, it follows that l_N is p-convex for some $p>2$ if and only if $l_{\tilde{N}}$ is p-convex for some $p>1$, hence if and only if $l_{\tilde{N}}$ is of type p for some $p>1$ ([22], p. 100), or what is the same, a B-convex space. In [7] it is shown that $l_{\tilde{N}}$ is B-convex if and only if both $\tilde{N}$ and its conjugate Orlicz function satisfy the Δ_2-condition, or what is the same (by [18], Theorem 4.2), condition (2.6). □

For example, if $N(t)=t^2/(1+|\log t|)$ then l_N is of type 2 but not p-convex for $p>2$; but if $N(t)=t^p/(1+|\log t|)$ or $N(t)=t^p(1+|\log t|)$, then l_N is p-convex, $p>2$.

Given a sequence $\{x_i\}\subset\mathbb{R}$, define $\{x_i^*\}$ as the non-increasing rearrangement of $\{|x_i|\}$. If $w=\{w_i\}$ is a non-increasing sequence of positive numbers, $w\in c_0-l_1$, then the Lorentz space $d(w,p)$, $p\geqq 1$, is defined as

$$d(w,p)=\{\{x_i\}\in\mathbb{R}^{\mathbb{N}}:\sum w_i(x_i^*)^p<\infty\} \quad \text{with} \quad \|\{x_i\}\|=(\sum w_i(x_i^*)^p)^{1/p}.$$

The sequence w is called regular if $\sum_{i=1}^{n} w_i\simeq n w_n$. $d(w,p)$ is always p-convex but not r-convex if $r>p$, and it is q-concave for some $q<\infty$ if and only if w is regular (see e.g. [31]). Hence:

2.5. Proposition. *$d(w,p)$ is of type 2 if and only if w is regular and $p\geqq 2$; and it is p-convex and q-concave (for some $q<\infty$) if and only if w is regular.*

2ii. The Limit Theorems

We start with a symmetrization lemma for triangular arrays which will also be useful in Sect. 6.

2.6. Lemma. *Let $\{X_{nj}: j=1,\dots,k_n,\ n\in\mathbb{N}\}$ be an infinitesimal triangular array of B-valued r.v.'s i.i.d. by rows, B a Banach space, and $\{\varepsilon_{nj}: j=1,\dots,k_n,\ n\in\mathbb{N}\}$ an array of row-wise independent Rademacher r.v.'s independent of $\{X_{nj}\}$. Then if $\left\{\mathscr{L}\left(\sum_{j=1}^{k_n}\varepsilon_{nj}(X_{nj}-a_{nj})\right)\right\}_{n=1}^{\infty}$ is tight for some $\{a_{nj}\}\subset B$, so is $\left\{\mathscr{L}\left(\sum_{j=1}^{k_n}(X_{nj}-EX_{nj}I_{[\|X_{nj}\|\leqq\delta]})\right)\right\}_{n=1}^{\infty}$ for every $\delta>0$; and if $\left\{\mathscr{L}\left(\sum_{j=1}^{k_n}X_{nj}\right)\right\}_{n=1}^{\infty}$ is shift tight, then $\left\{\mathscr{L}\left(\sum_{j=1}^{k_n}\varepsilon_{nj}(X_{nj}-EX_{nj}I_{[\|X_{nj}\|<\delta]})\right)\right\}_{n=1}^{\infty}$ is tight for every $\delta>0$.*

Proof. The second part is just Lemma 3.5 in [2] (Ex. 3.8.11 in [5]). For the first part, we apply Propositions 2.5, 3.2 and 3.11 in [2] (3.5.6, 3.4.8, 3.4.9 in [5]): if $\mu_{nj}=\mathscr{L}(X_{nj}-a_{nj})$ and $\bar{\mu}_{nj}$ is the conjugate measure, then $\mathscr{L}(\varepsilon_{nj}(X_{nj}-a_{nj}))=\frac{1}{2}(\mu_{nj}+\bar{\mu}_{nj})$ and the following implications follow (in the notation of [2] and [5])

$$\begin{aligned}&\{\mathscr{L}(\sum_j\varepsilon_{nj}(X_{nj}-a_{nj}))\}\ \text{tight}\ \Rightarrow\ \{\text{Pois}\ \tfrac{1}{2}\sum_j(\mu_{nj}+\bar{\mu}_{nj})\}\ \text{tight}\\ &\Rightarrow\ \{\text{Pois}\sum_j\mu_{nj}\}\ \text{shift tight}\ \Rightarrow\ \{\mathscr{L}(\sum_j X_{nj})\}\ \text{shift tight}\\ &\Rightarrow\ \{\mathscr{L}(\sum_j(X_{nj}-EX_{nj}I_{[\|X_{nj}\|\leqq\delta]}))\}\ \text{tight for every}\ \delta>0.\quad\square\end{aligned}$$

2.7. Corollary. *Let X be a centered B-valued r.v. and ε a Rademacher r.v. independent of X. Then, $X\in\mathrm{CLT}\Leftrightarrow\varepsilon X\in\mathrm{CLT}$.*

Proof. If either X or εX satisfy the CLT then $nP\{\|X\|>n^{\frac{1}{2}}\}\to 0$ and therefore (direct computation) the centerings $\{n^{\frac{1}{2}}EXI_{[\|X\|\leqq n^{\frac{1}{2}}]}\}$ tend to zero. Also, $X\in$CLT if and only if $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/n^{\frac{1}{2}}\right)\right\}$ is tight, and likewise for εX. These two observations together with the previous lemma give the result. □

We also need the next lemma.

2.8. Lemma. *For any $c>0$, if $X\in$CLT and X is symmetric, then also $XI_{[\|X\|\leqq c]}\in$CLT.*

Proof. The lemma follows directly from the following inequality: if X_i are independent copies of X and K is convex and symmetric, then ([16]; [5], Lemma 3.5.1):

$$P\left\{\sum_{i=1}^{n} X_i I_{[\|X_i\|\leqq c]}\in K^c\right\}\leqq 2P\left\{\sum_{i=1}^{n} X_i\in K^c\right\} \tag{2.7}$$

for all $n=1,\ldots,$. □

The following theorem relates CLT and WLLN in 2-convex Banach lattices (i.e., in those Banach lattices where taking squares makes sense).

2.9. Theorem. *Let B be a 2-convex Banach lattice. Then, for B-valued r.v.'s X we have:*

1) $X\in$CLT (*in* B) $\Rightarrow |X|^2\in$WLLN (*in* $B_{(2)}$);

2) *B is of type 2 if and only if* $|X|^2\in$WLLN (*in* $B_{(2)}$) $\Rightarrow X\in$CLT (*in* B).

Proof. Since εX is symmetric, $|\varepsilon X|=|X|$ and εX and X both have the same covariance, by Corollary 2.7 it is enough to prove this theorem for X symmetric. Let X_i be independent copies of X symmetric and denote $X_i I_{[\|X_i\|\leqq c]}$ by X_{ic}. Assume $X\in$CLT. By Lemma 2.8, $XI_{[\|X\|\leqq c]}\in$CLT and therefore so does $XI_{[\|X\|>c]}$ for all $c>0$. If γ^c is the centered Gaussian p.m. with the covariance of X_i-X_{ic}, then $\gamma^c\to\delta_0$ as $c\uparrow\infty$ (note that γ^c is a convolution factor of γ and that its covariance tends to zero). We then conclude by [3], Theorem 5.1 (or [5], 3.6.18) that

$$\lim_{c\uparrow\infty}\lim_{n} E\left\|\sum_{i=1}^{n}(X_i-X_{ic})/n^{\frac{1}{2}}\right\|=\lim_{c\uparrow\infty}\int\|x\|\,d\gamma^c(x)=0. \tag{2.8}$$

Since $|X_i-X_{ic}|^2=|X_i|^2-|X_{ic}|^2(=|X_i|^2 I_{[\|X\|>c]})$, the left side inequality in (2.4) gives (by (2.2)) that

$$\lim_{c\uparrow\infty}\lim_{n} E\left\|\left\|\sum_{i=1}^{n}|X_i|^2/n-\sum_{i=1}^{n}|X_{ic}|^2/n\right\|\right\|^{\frac{1}{2}}=0. \tag{2.9}$$

The $|X_{ic}|^2$ being bounded, the Mourier law of large numbers ([27]) implies that

$$\sum_{i=1}^{n}|X_{ic}|^2/n\to E|X|^2 I_{[\|X\|\leqq c]} \quad \text{a.s. in } B_{(2)}. \tag{2.10}$$

Now, (2.9) and (2.10) give that the sequence $\left\{\sum_{i=1}^{n} |X_i|^2/n\right\}$ is Cauchy in probability, therefore that

$$E|X|^2 = \lim_{c\uparrow\infty} E|X|^2 I_{[\|X\| \leq c]} \quad \text{exists and} \tag{2.11}$$

$$\lim_n \sum_{i=1}^{n} |X_i|^2/n = E|X|^2 \quad \text{in probability.} \tag{2.12}$$

Assume now that $|X|^2 \in \mathrm{WLLN}$ in $B_{(2)}$ implies that $X \in \mathrm{CLT}$. By Mourier's law of large numbers we have that $E\|X\|^2 < \infty \Rightarrow |X|^2 \in \mathrm{SLLN} \Rightarrow X \in \mathrm{CLT}$, and therefore ([14]) B is of type 2.

Finally, let us assume B of type 2 and $|X|^2 \in \mathrm{WLLN}$ (X symmetric). Then by Theorem 2.10 in [2] ([5], 3.5.9), $E|X|^2$ exists as the limit in (2.11) and (2.12) holds true. This, together with (2.10) implies by uniform integrability that

$$\lim_{c\uparrow\infty} \lim_n E \left|\left|\left| \sum_{i=1}^{n} |X_i|^2/n - \sum_{i=1}^{n} |X_{i_c}|^2/n \right|\right|\right|^{\frac{1}{2}}$$
$$= \lim_{c\uparrow\infty} |||E|X|^2 I_{[\|X\| > c]}|||^{\frac{1}{2}} = 0.$$

Now the right side of Hinčin-Maurey's inequality (2.4) gives the relation (2.8). Since $X_{i_c} \in \mathrm{CLT}$ by the Hoffmann-Jørgensen and Pisier CLT ([14]), it follows that $X \in \mathrm{CLT}$ by an observation of Pisier ([28], Theorem 3.1, first part; [5], Ex. 1.4.12). □

This result admits an extension to triangular arrays and general infinitely divisible limits, at least if B has an unconditional basis, just as done for $B = \mathbb{R}$ in [10]. If B has an unconditional basis $\{e_i\}$ then, as mentioned in 2i, it is a Banach lattice (with the natural order and after some renorming if necessary). Special features of such a Banach lattice are that if $\{Y_i\}$ are independent, symmetric B-valued r.v.'s, then

$$E|\sum_i Y_i|^2 = \sum_i E|Y_i|^2$$

(which is easy to obtain coordinatewise), and that the transformation $x \to |x|^2$ commutes with the projections $\sum x_i e_i \to \sum_{i\in I} x_i e_i$, $I \subset \mathbb{N}$. Aside from the use of these two facts together with the use of results in [2] or [5] instead of the classical CLT results for the line, the proof of the following theorem does not differ significantly from the proof of the corresponding one for the line in [10]. So, we omit the proof. In the next theorem, the notation is as in [2] or [5], except that in $B_{(2)}$ truncations are considered with respect to $|||\phi(x)|||_0 = \|x\|^2$, so that for instance the p.m. c_τ Pois ν has characteristic function

$$(c_\tau \operatorname{Pois} \nu)^\wedge(f) = \exp\{\int (e^{i\langle f, x\rangle} - 1 - i\langle f, x\rangle I_{[|||y|||_0 \leq \tau]}(y))\, d\nu(y), \qquad f \in B'.$$

We also let $T(x) = \phi(|x|) = |x|^2$, $x \in B$.

2.10. Theorem. *Let B be a type 2 Banach space with an unconditional basis. Let $\{X_{nj}: j=1, \ldots, k_n, n \in \mathbb{N}\}$ be a symmetric infinitesimal array of row wise independent B-valued r.v.'s such that for all f in a w^*-sequentially dense subset of B',*

$$\mathscr{L}(\sum_j \langle f, X_{nj}\rangle) \to_w \gamma_f * \operatorname{Pois} \mu_f, \tag{2.13}$$

where γ_f and μ_f are symmetric Gaussian and Lévy measures on $\mathbb{R}$ respectively. Then:

a) *If there exist γ Gaussian and μ Lévy measures on B such that*

$$\mathscr{L}(\sum_j X_{nj}) \to_w \gamma * \operatorname{Pois} \mu \quad \text{in } B, \tag{2.14}$$

then $\mu \circ T^{-1}$ is a Lévy measure on $B_{(2)}$ and

$$\mathscr{L}(\sum_j |X_{nj}|^2) \to_w \delta_{a_\tau} * c_\tau \operatorname{Pois}(\mu \circ T^{-1}) \quad \text{in } B_{(2)} \tag{2.15}$$

for all $\tau > 0$ such that $\mu\{\|x\| = \tau^{\frac{1}{2}}\} = 0$, where

$$a_\tau = \int |x|^2 \, d(\gamma * \operatorname{Pois}(\mu|_{[\|x\| \leq \tau^{\frac{1}{2}}]})(x).$$

Moreover, for these values of τ,

$$\sum_j E|X_{nj}|^2 I_{[\|X_{nj}\| \leq \tau^{\frac{1}{2}}]} \to a_\tau. \tag{2.16}$$

b) *If $\{\mathscr{L}(\sum_j |X_{nj}|^2)\}_{n=1}^\infty$ is tight, then (2.14) holds for measures γ and μ determined by γ_f and μ_f. Therefore, (2.15) and (2.16) hold too.*

Remark. Without the existence of an unconditional basis it is possible to prove that if $\mathscr{L}(\sum_j |X_{nj}|^2) \to_w \operatorname{Pois}(\mu \circ T^{-1})$, μ a symmetric Lévy measure on B, then $\mathscr{L}(\sum_j X_{nj}) \to_w \operatorname{Pois} \mu$, and conversely. This is done using Theorem 2.10 in [24] (or the desymmetrized version of this theorem in [5], Ex. 3.5.5).

3. Estimating $E\|S_n\|^p$ in Banach Spaces

Let $\{X_i\}_{i=1}^\infty$ be independent B-valued r.v.'s, $S_n = \sum_{i=1}^n X_i$, $n=1, \ldots$, and $p>0$. In this section we give equivalent (up to constants) upper and lower bounds for $E\|S_n\|^p$. This will allow us to obtain results on the CLT in some $l_2(B)$ spaces where the conditions of Theorem 4.3 fail (see Sect. 5). These bounds have independent interest and may be useful in other situations as well (see e.g. Sect. 5 and 6).

The problem of estimating $E\|S_n\|^p$ in terms of $\mathscr{L}(X_1)$ in the i.i.d. case has been solved with remarkable accuracy by Klass [17] in the line, and by Kuelbs and Zinn [20] in Hilbert spaces. The estimations given here apply in more general situations, and in the cases considered by these authors, are equivalent to their bounds up to constants (ours are less precise, but possibly easier to handle). The proof is mainly based on Hoffmann-Jørgensen's upper bound of $E\|S_n\|^p$ in terms of $E\max_{i\le n}\|X_i\|^p$ and the quantiles of $\|S_n\|$. We recall it for ease of reference:

3.1. Lemma. *Let $\{X_i\}_{i=1}^\infty$ be independent, symmetric B-valued r.v.'s, $X_i\in L_p(P)$ for some (fixed) $p>0$. Then*

$$E\left\|\sum_{i=1}^\infty X_i\right\|^p \leqq 2\cdot 3^p E\sup_i \|X_i\|^p + 8\cdot 3^p t_0^p \tag{3.1}$$

where $t_0=\inf\left[t>0\colon P\left\{\left\|\sum_{i=1}^\infty X_i\right\|>t\right\}\leqq 1/8\cdot 3^p\right]$.

For the proof see [13], p. 164–165 (or [5], p. 107).

The next lemma gives estimates of $E\sup_i\|X_i\|^p$ in terms of the marginal distributions of the X_i.

3.2. Lemma. *Let $\{\xi_j\}_{j=1}^\infty$ be independent real r.v.'s, $\xi_j\in L_p(P)$, $p>0$. Given $\lambda>0$, let $\delta_0=\inf[t>0\colon \sum_j P\{|\xi_j|>t\}\leqq\lambda]$. Then,*

$$\begin{aligned}\lambda(1+\lambda)^{-1}\delta_0^p+p(1+\lambda)^{-1}\sum_{j=1}^\infty\int_{\delta_0}^\infty t^{p-1}P\{|\xi_j|>t\}\,dt\\ \leqq E\sup_j|\xi_j|^p\leqq\delta_0^p+p\sum_{j=1}^\infty\int_{\delta_0}^\infty t^{p-1}P\{|\xi_j|>t\}\,dt.\end{aligned}\tag{3.2}$$

Proof. Since

$$\begin{aligned}P\{\sup_j|\xi_j|>t\}&=1-\prod_{j=1}^\infty[1-P\{|\xi_j|>t\}]\geqq 1-\exp[-\textstyle\sum_j P\{|\xi_j|>t\}]\\ &\geqq\textstyle\sum_j P\{|\xi_j|>t\}/(1+\sum_j P\{|\xi_j|>t\})\\ &\geqq\begin{cases}(1+\lambda)^{-1}\sum_j P\{|\xi_j|>t\} & \text{if } t>\delta_0\\ \lambda/(1+\lambda) & \text{if } t\leqq\delta_0,\end{cases}\end{aligned}$$

it follows that

$$\begin{aligned}E\sup_j|\xi_j|^p&=p\int_0^\infty t^{p-1}P\{\sup_j|\xi_j|>t\}\,dt\\ &=p\left(\int_0^{\delta_0}+\int_{\delta_0}^\infty\right)t^{p-1}P\{\sup_j|\xi_j|>t\}\,dt\\ &\geqq\lambda(1+\lambda)^{-1}\delta_0^p+p(1+\lambda)^{-1}\sum_j\int_{\delta_0}^\infty t^{p-1}P\{|\xi|_j>t\}\,dt.\end{aligned}$$

On the other hand,

$$E\sup_j|\xi_j|^p \leqq p\int_0^{\delta_0} t^{p-1} P\{\sup_j|\xi_j|>t\}\,dt + p\int_{\delta_0}^{\infty} t^{p-1}\sum_j P\{|\xi_j|>t\}\,dt$$

$$\leqq \delta_0^p + p\sum_j \int_{\delta_0}^{\infty} t^{p-1} P\{|\xi_j|>t\}\,dt. \quad \square$$

The estimations of $E\|S_n\|^p$ are as follows:

3.3. Theorem. *Fix $p, r>0$. Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of independent, symmetric B-valued r.v.'s such that $E\|X_j\|^p<\infty$. Let $\delta_0=\inf[t: \sum_j P\{\|X_j\|>t\}\leqq 1/8\cdot 3^{p\vee r}]$. Then there exist constants C_1, $C_2\in(0,\infty)$, depending only on p and r such that*

$$C_1\left[E\sup_j\|X_j\|^p + \left(E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^r\right)^{p/r}\right.$$
$$\leqq E\left\|\sum_{j=1}^{\infty} X_j\right\|^p \leqq C_2\left[E\sup_j\|X_j\|^p + \left(E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^r\right)^{p/r}\right]. \tag{3.3}$$

Remark. Note that, in view of Lemma 3.2, (3.3) gives an upper bound for $E\left\|\sum_{j=1}^{\infty} X_j\right\|^p$, $p>0$, in terms of only the distributions of the individual X_j if B is of type r, and a lower bound if B is of cotype r. This consequence of Theorem 3.3 is essentially another (perhaps more convenient) form of Lemma 1 in [20] in the case of i.i.d. summands. Let us state its immediate corollary for Hilbert spaces.

3.4. Corollary. *Let B be a Hilbert space in Theorem 3.3. Then there exist constants C_1, $C_2\in(0,\infty)$ depending only on $p>0$ such that*

$$C_1\left[E\sup_j\|X_j\|^p + \left(\sum_{j=1}^{\infty} E\|X_j I_{[\|X_j\|\leqq\delta_0]}\|^2\right)^{p/2}\right]$$
$$\leqq E\left\|\sum_{j=1}^{\infty} X_j\right\|^p \leqq C_2\left[E\sup_j\|X_j\|^p + \left(\sum_{j=1}^{\infty} E\|X_j I_{[\|X_j\|\leqq\delta_0]}\|^2\right)^{p/2}\right]. \tag{3.4}$$

Proof of Theorem 3.3. By the triangle inequality,

$$E\left\|\sum_{j=1}^{\infty} X_j\right\|^p \leqq c_p E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^p + c_p E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|>\delta_0]}\right\|^p. \tag{3.5}$$

Let us first consider the second summand on the right side of (3.5). By (3.1),

$$E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|>\delta_0]}\right\|^p \leqq 2\cdot 3^p E\sup_j\|X_j\|\, I_{[\|X_j\|>\delta_0]} + 8\cdot 3^p t_0^p. \tag{3.6}$$

In order to find t_0 let us note that

$$P\left\{\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|>\delta_0]}\right\|>0\right\} \leqq \sum_{j=1}^{\infty} P\{\|X_j\|>\delta_0\} \leqq 1/8\cdot 3^p.$$

Hence $t_0=0$ and

$$E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|>\delta_0]}\right\|^p \leqq 2\cdot 3^p E\sup_j \|X_j\|^p. \tag{3.7}$$

Next we bound the first summand. Since

$$P\left\{\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|>t\right\}\leqq t^{-r} E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^r,$$

inequality (3.1) gives:

$$E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^p \leqq 2\cdot 3^p E\sup_j \|X_j\|^p I_{[\|X_j\|\leqq\delta_0]} + 8\cdot 3^p\left(8\cdot 3^p E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^r\right)^{p/r}. \tag{3.8}$$

(3.7) and (3.8), on account of (3.5), give the right side inequality in (3.3).

To prove the left side, let us note that by Lévy's inequality,

$$E\sup_j \|X_j\|^p \leqq 2E\left\|\sum_{j=1}^{\infty} X_j\right\|^p. \tag{3.9}$$

Let us also note that by Lemma 3.5.1 in [5] (see also [16], or (2.7) above)

$$E\left\|\sum_{j=1}^{\infty} X_j\right\|^p \geqq 4^{-1}\left[E\left\|\sum_{j=1} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^p + E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|>\delta_0]}\right\|^p\right]. \tag{3.10}$$

Now, if $p\geqq r$, Hölder's inequality gives

$$E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^p \geqq \left[E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^r\right]^{p/r}, \tag{3.11}$$

and the left side of (3.3) follows from (3.9)–(3.11). The case $p\leqq r$ requires another application of (3.1) and (3.2). The positive numbers p and r in inequality (3.8) are interchangeable, hence

$$E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^r \leqq 2\cdot 3^r \delta_0^r + 8\cdot 3^r\left(8\cdot 3^r E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^p\right)^{r/p}. \tag{3.12}$$

Raising both sides of this inequality to the p/r-th power, and applying (3.2), (3.9) and (3.10), we get

$$\left(E\left\|\sum_{j=1}^{\infty} X_j I_{[\|X_j\|\leqq\delta_0]}\right\|^r\right)^{p/r} \leqq cE\left\|\sum_{j=1}^{\infty} X_j\right\|^p. \tag{3.13}$$

Now the left side of inequality (3.3) follows from (3.9) and (3.13). □

4. On the CLT in [29]

Theorem 2.9 provides a simple proof of the CLT in [29], Theorem 5.1. In order to provide some additional information on that theorem, we will prove first the following two lemmas.

4.1. Lemma. *The maps $x \to |x|^2$ from B into $B_{(2)}$ and $|x|^2 \to |x|$ from the positive cone of $B_{(2)}$ into B are continuous.*

Proof. $||x|-|y|| \leqq ||x|^2-|y|^2|^{\frac{1}{2}}$ because the same is true for functions. Hence $\| |x|-|y| \| \leqq c^{-1} \||\, |x|^2-|y|^2 \||^{\frac{1}{2}}$ (c as in (2.2)). Conversely, using the Hölder type inequalities in [22], p. 43, we get

$$\||\, |x|^2-|y|^2 \|| \leqq c \| \, ||x|^2-|y|^2|^{\frac{1}{2}} \|^2$$
$$= c \| \, ||x|-|y||^{\frac{1}{2}} \, ||x|+|y||^{\frac{1}{2}} \|^2 \leqq c \| \, |x|-|y| \, \| \, \| \, |x|+|y| \, \|. \quad \square$$

4.2. Lemma. *Let B be a 2-convex Banach lattice and let X be a pregaussian B-valued r.v. Then $E|X|^2$ exists (as defined in (2.11)) and*

$$E|X|^2 = \int |x|^2 \, d\gamma(x)$$

where γ is the centered Gaussian p.m. with the covariance of X.

Proof. If ε is Rademacher independent of X, then εX has the same covariance as X and $|\varepsilon X| = |X|$. So, we may and do assume that X is symmetric. Assume X is pregaussian and let $X_n = X I_{[\|X\| \leqq n]}$. Then there exist G_n, G Gaussian r.v.'s with the covariances of X_n, X respectively such that $G_n \to G$ a.s. and in L_p for all $p < \infty$ (note that the distribution of G_n is a convolution factor of that of G and converges to it as $n \to \infty$). Therefore Lemma 4.1 and the boundedness of the moments of the norm give

$$E|G_n|^2 \to E|G|^2, \tag{4.1}$$

where expectation is in Bochner's sense. Now, $|X_n|^2$ being bounded, it is Bochner integrable for all n (although $|X|^2$ may not be). So, by (4.1) and (2.11), the lemma will be proved if we show that $E|X_n|^2 = E|G_n|^2$. For $f \in (B_{(2)})'$, $f \geqq 0$, define the following real function on B:

$$x \to \|x\|_f = (f(|x|^2))^{\frac{1}{2}}.$$

Then ([19], Proposition 1) $\| \cdot \|_f$ is a seminorm, and the space $(B/\| \cdot \|_f, \| \cdot \|_f)$ is pre-Hilbert (and separable because B is). By Lemma 4.1 the projection map $\pi_f \colon B \to B/\| \cdot \|_f$ is continuous. Hence, $\pi_f X_n$ is a pregaussian r.v. with values in the Hilbert space $(B/\| \cdot \|_f, \| \cdot \|_f)^-$ (the completion of $B/\| \cdot \|_f$). Since G_n is a Gaussian r.v. with the covariance of X_n, then $\pi_f X_n$ has the covariance of the Gaussian r.v. $\pi_f G_n$ and therefore

$$E \|\pi_f X_n\|_f^2 = E \|\pi_f G_n\|_f^2$$

or what is the same, $Ef(|X_n|^2) = Ef(|G_n|^2)$. Since every $f \in (B_{(2)})'$ is the difference of two non-negative linear functionals, this identity holds true for every $f \in (B_{(2)})'$. Therefore, $E|X_n|^2 = E|G_n|^2$. □

Theorem 5.1 in [29], in full generality, is as follows:

4.3. Theorem. *Let B be a p-convex, q-concave Banach lattice, $p>2$, $q<\infty$. Then the following are equivalent for B-valued r.v.'s X:*

1) $X \in \mathrm{CLT}$;

2) *X is pregaussian and* $\lim_n nP\{\|X\| > n^{\frac{1}{2}}\} = 0$;

3) $E|X|^2$ *exists (in the sense that* $E|X|^2 = \lim_n E|X|^2 I_{[\|X\| \le n]}$) *and* $\lim_n nP\{\|X\| > n^{\frac{1}{2}}\} = 0$.

Proof. If $X \in \mathrm{CLT}$, then it is obviously pregaussian and it has been shown in the proof of Theorem 2.9 (or in Lemma 4.2) that $E|X|^2$ exists; moreover, $nP\{\|X\| > n^{\frac{1}{2}}\} \to 0$ ([29], Proposition 5.2; [5] 3.6.21). By Lemma 4.2, if X is pregaussian then $E|X|^2$ exists. So, we need only prove that 3) implies 1). Let X' be an independent copy of X. Then the tail condition in 3) implies that

$$nP\{\| |X|^2 - |X'|^2 \| > n\} \to 0,$$

hence, by [23], Theorem 4.1 (or [24], Corollary 3.3), that $|X|^2 - |X'|^2 \in \mathrm{WLLN}$ (notice that $B_{(2)}$ is of type $\min(p/2, 2) > 1$: see e.g. [22]). Therefore,

$$\left\{\mathscr{L}\left(\sum_{i=1}^n |X_i|^2/n - E|X|^2 I_{[\|X\| \le n^{\frac{1}{2}}]}\right)\right]_{n=1}^{\infty}$$

is tight with only degenerate, centered limits (see e.g. [2], 2.10 for the centering), i.e.

$$\sum_{i=1}^n |X_i|^2/n - E|X|^2 I_{[\|X\| \le n^{\frac{1}{2}}]} \to_{pr} 0.$$

This implies, by the conditions in 3), that $|X|^2 \in \mathrm{WLLN}$. So, $X \in \mathrm{CLT}$ by Theorem 2.9. □

Remark. Note that the proof that in type $p>1$ the tail condition $nP\{\|X\| > n\} \to 0$ implies the weak law is very easy; hence Theorem 2.9 effectively reduces Theorem 4.3 to a more elementary proposition.

But Theorem 4.3 does not generalize to all type 2 Banach spaces. As a matter of fact we have:

4.4. Theorem. *For a Banach space B the following are equivalent:*

1) *B is of cotype* 2,

2) (*i*) $X \in l_2(B)$,

(ii) *X is pregaussian,*

(iii) $\lim_n nP\{\|X\| > n^{\frac{1}{2}}\} = 0$,

imply $X \in \mathrm{CLT}$.

Remark. A forerunner of this theorem is a result of S. Kwapien (private communication) which shows that there do not exist $r>2$ and $c<\infty$ such that for all independent centered $Y_j \in l_1(l_p)$ $(p>2)$ which are pregaussian (with associated independent Gaussian r.v.'s G_j), and in L_r, $E\|\sum_j Y_j\|^r \leqq c[\sum_j E\|Y_j\|^r + E\|\sum_j G_j\|^r]$.

Before proving Theorem 3.4 it is convenient to observe the following:

4.5. Lemma. *Let B be a Banach space which is not of cotype 2. Then for any sequence $c_n \downarrow 0$ there exist symmetric, independent, pregaussian B-valued r.v.'s Y_n with associated (symmetric) Gaussian r.v.'s G_n, such that:*

1) $\|Y_n\|=1$ *a.s.*
2) $E\|G_n\|^2 \leqq c_n$.

Proof. The cotype 2 property is equivalent to the fact that there exists $c<\infty$ such that $E\|Y_G\|^2 \leqq cE\|G\|^2$ for every symmetric Gaussian B-valued r.v. and weakly centered pregaussian Y_G with the covariance of G (see e.g. [3], Theorem 5.2). Hence we can find X_n, G_n, centered, with the same covariance, and G_n Gaussian such that

$$E\|X_n\|^2 \leqq 1 \quad \text{and} \quad E\|G_n\|^2 \leqq c_n.$$

By dividing by $(E\|X_n\|^2)^{\frac{1}{2}}$, we may as well assume $E\|X_n\|^2=1$. Consider now new probability spaces $(\Omega, \mathscr{F}, Q_n)$ ($(\Omega, \mathscr{F}, P)$ is the space where the X_n and G_n are defined), with $dQ_n(\omega)=\|X_n(\omega)\|^2 dP(\omega)$, and on each of them a random variable $Y_n(\omega)=X_n(\omega)/\|X_n(\omega)\|$. Then $\|Y_n(\omega)\|=1$ and Y_n is G_n-pregaussian (as for any $f, g \in B'$, $\int f(Y_n)g(Y_n)dQ_n=\int f(X_n)g(X_n)dP=\int f(G_n)g(G_n)dP$). Now we can take all the random variables G_n, Y_n defined on the same probability space $(\Omega, \mathscr{F}, P)\times(\Omega, \mathscr{F}, Q_1)\times \ldots \times(\Omega, \mathscr{F}, Q_n)\times \ldots$. □

Proof of Theorem 4.4. 1) ⇒ 2). If B is of cotype 2, then $l_2(B)$ is also of cotype 2, obviously. But in cotype 2 spaces X pregaussian is necessary and sufficient for $X \in \mathrm{CLT}$ ([16]).

2) ⇒ 1). By Lemma 4.5, for any $r_\alpha \to 0$ $(\alpha \in \mathbb{N})$ there exist independent symmetric B-valued r.v.'s Y_α such that Y_α is pregaussian, and if G_α are the associated symmetric Gaussian r.v.'s then

$$\sum_{\alpha=1}^{\infty} r_\alpha^{\frac{1}{2}} E\|G_\alpha\| < \infty \quad \text{and} \quad \|Y_\alpha\|=1 \quad \text{a.s.} \tag{4.2}$$

Let N be an integer-valued r.v. such that $P\{N=m\}=c/m^2 \ln_2 m$ (for $m>0$ sufficiently large), where we write $\ln_2 m$ for $\ln(\ln m)$. Take $r_\alpha = P\{N^2 \leqq \alpha < N^2+N\}$, and let Y_α, G_α, $\alpha=1, \ldots$, be a set of independent B-valued r.v.'s satisfying (4.2) for these r_α, with the Y_α independent of N. Then, define the following $l_2(B)$-valued r.v.:

$$X = \sum_{N^2 \leqq \alpha < N^2+N} Y_\alpha e_\alpha.$$

Since $\|X\|^2 = N$ ($\|x\|$ will denote the norm in $l_2(B)$ or in B depending on whether $x \in l_2(B)$ or $x \in B$), it follows that

$$\lim_n nP\{\|X\| > n^{\frac{1}{2}}\} = 0. \tag{4.3}$$

If we put $G = \sum_\alpha r_\alpha^{\frac{1}{2}} G_\alpha e_\alpha$, then, by (4.2), G is a well defined Gaussian $l_2(B)$-valued r.v. Moreover, if $f = (f_\alpha)$, $g = (g_\alpha)$ are in $l_2(B') = (l_2(B))'$, then we have

$$\begin{aligned} E\langle f, X\rangle\langle g, X\rangle &= E \sum_{\alpha, \beta} \langle f_\alpha, Y_\alpha\rangle \langle g_\beta, Y_\beta\rangle I_{[N^2 \leqq \alpha, \beta < N^2 + N]} \\ &= \sum_\alpha E\langle f_\alpha, Y_\alpha\rangle\langle g_\alpha, Y_\alpha\rangle P\{N^2 \leqq \alpha < N^2 + N\} \\ &= \sum_\alpha E\langle f_\alpha, G_\alpha\rangle\langle g_\alpha, G_\alpha\rangle r_\alpha = E\langle f, G\rangle\langle g, G\rangle. \end{aligned}$$

Hence, X is pregaussian in $l_2(B)$ with associated Gaussian G. This and (4.3) mean that X satisfies 4.4 (2(i), (ii), (iii)).

The theorem will be proved if we show that $X \notin \mathrm{CLT}$ (in $l_2(B)$). For this it is enough to prove that

$$E\left\| n^{-\frac{1}{2}} \sum_{j=1}^n X_j I_{[\|X_j\| \leqq n^{\frac{1}{2}}]}\right\|^2 \to \infty \quad \text{as } n \to \infty. \tag{4.4}$$

(If (4.4) holds, then by Lemma 3.1 it follows that the sequence $\left\{n^{-\frac{1}{2}} \sum_{j=1}^n X_j I_{[\|X_j\| \leqq n^{\frac{1}{2}}]}\right\}$ is not stochastically bounded, and this implies, by (2.7), that neither is $\{\sum_j X_j/n^{\frac{1}{2}}\}$).

Let $\{N_j, Y_{\alpha,j}, \alpha = 1, \ldots\}_{j=1}^\infty$ be independent copies of $(N, Y_\alpha, \alpha = 1, \ldots)$, let X_j be independent copies of X defined with these N_j and $Y_{\alpha,j}$, let $T_n = \sum_j X_j I_{[\|X_j\| \leqq n^{\frac{1}{2}}]}$ and $I_{\alpha,j}^n = I_{[N_j^2 \leqq \alpha < N_j^2 + N_j, N_j \leqq n]}$. We then have

$$\begin{aligned} E\|T_n\|^2 &= \sum_\alpha E\left\|\sum_{j=1}^n Y_{\alpha,j} I_{\alpha,j}^n\right\|^2 \\ &\geqq \tfrac{1}{2} \sum_\alpha E \max_{j \leqq n} \|Y_{\alpha,j} I_{\alpha,j}^n\|^2 \quad \text{(by Lévy's inequality)} \\ &= \tfrac{1}{2} \sum_\alpha E \max_{j \leqq n} I_{\alpha,j}^n \quad \text{(by (4.2))} \\ &\geqq \tfrac{1}{2} \sum_{k=1}^n k E \max_{j \leqq n} I_{[N_j = k]} \\ &\geqq \tfrac{1}{4} \sum_{[k \leqq n:\, nP\{N=k\} \leqq 1]} kn \int_0^1 P\{N = k\}\, dt \quad \text{(by Lemma 3.2)} \\ &\geqq \tfrac{1}{4} n \sum_{d(n/\ln_2 n)^{\frac{1}{2}} \leqq k \leqq n} c/k \ln_2 k \quad \text{(for some } d > 0\text{, by direct computation)} \\ &\geqq \eta n \ln n/\ln_2 n \quad \text{(for some } \eta > 0\text{)}. \end{aligned}$$

Hence,

$$E\|T_n/n^{\frac{1}{2}}\|^2 \geqq \eta \ln n/\ln_2 n \to \infty,$$

i.e. (4.4). □

It would be interesting to determine exactly in what type 2 Banach spaces (or at least type 2 Banach lattices) Theorem 4.3 holds. While the question is now settled for spaces of the form $l_2(B)$, the question remains open in general. In this connection it may be interesting to note that there is an example of a type 2 space B with an unconditional basis, which is of cotype p for all $p>2$, and such that l_2 is not isomorphic to any subspace of any quotient of B ([15], Example 2.2).

Theorem 4.4 implies:

4.5. Corollary. *There exists in $l_2(l_p)$, for every $p>2$, a r.v. X such that $nP\{\|X\|>n^{\frac{1}{2}}\}\to 0$, X is pregaussian, but $X\notin$CLT.*

In fact, a similar proof gives a stronger statement:

4.6. Corollary. *There exists in $l_1(l_r)$, for every $r>1$, a r.v. Y such that $nP\{\|Y\|>n\}\to 0$, $E|Y|$ exists (in the sense of (2.11)), but $Y\notin$WLLN.*

Remark. Necessary and sufficient conditions for the CLT in $l_2(l_p)$ for $p>2$, and the WLLN in $l_1(l_r)$, $r>1$, will be given in the next section.

5. The WLLN in l_1 and $l_1(l_p)$ and the CLT in $l_2(l_p)$

Corollary 4.5 raises the question of finding necessary and sufficient conditions for $X\in$CLT in $l_2(l_p)$, $p>2$. This question is less esoteric than it looks at first sight: in view of Theorems 4.3 and 4.4, these spaces are the next "natural" family of Banach spaces where it may be possible to find n.a.s.c. for the CLT, and such conditions must necessarily be different from the conditions for l_p. It turns out that Theorem 3.3 (together with Theorem 2.9) is useful in solving this problem. An interesting problem which we will solve first is the WLLN in l_1.

5i. The Weak Law of Large Numbers in l_1

If B is a Banach space of (Rademacher) type p, $p>1$ (or equivalently a B-space, or equivalently a type 1-stable Banach space; see e.g. [24], p. 73–74, for the definitions) then it is well known ([23], Theorem 4.1; [24], Corollary 3.3) that a symmetric B-valued r.v. X satisfies the WLLN if and only if $nP\{\|X\|>n\}\to 0$. The following is a non-symmetric version of this result.

5.1. Theorem. *Let B be a (Rademacher) type p Banach space, $p>1$, and let X be a B-valued r.v. Then $X\in$WLLN if and only if:*

1) $nP\{\|X\|>n\}\to 0$ *as* $n\to\infty$,
2) $EX=\lim_n EXI_{[\|X\|\leqq n]}$ *exists.*

And then, $\sum_{i=1}^{n} X_i/n \to_{pr} EX$.

Proof. If 1) and 2) hold, then $\sum_{i=1}^{n} f(X_i)/n \to_{pr} f(EX)$ for all $f \in B'$ (as $nP\{|f(X)|>n\}\to 0$, $n^{-1}Ef^2(X)I_{[|f(X)|\leq n]}\to 0$ and $Ef(X)I_{[|f(X)|\leq n]}\to f(EX)$). 1) implies that $\varepsilon X \in$ WLLN by the above mentioned result, and this and 2) imply by Lemma 2.6 that $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/n\right)\right\}_{n=1}^{\infty}$ is tight, hence that it tends to EX. Conversely, if $X \in$ WLLN, then 1) and 2) follow from the general converse CLT (2.10, [2]). □

This theorem does not apply to l_1. We will use Theorem 3.3 to derive a set of n.a.s.c. for the WLLN in l_1. The following well known lemma is also needed:

5.2. Lemma. *Let X be a symmetric B-valued r.v., B a Banach space, and let X_i, $i=1,\ldots$, be independent copies of X. Then $X \in$ WLLN if and only if:*

1) $nP\{\|X\|>n\}\to 0$ *as* $n\to\infty$,

2) $n^{-1}E\left\|\sum_{i=1}^{n} X_i^n\right\|\to 0$ *as* $n\to\infty$, *where* $X_i^n = X_i I_{[\|X_i\|\leq n]}$.

Direct proofs of this lemma are easy to produce, but we will not do so because it is a direct consequence of 2.3 and 2.10, 2.14 in [2] (or 3.5.7, 9 in [5]).

5.3. Theorem. *Let $X=\sum_{\alpha} X_\alpha e_\alpha$ be a l_1-valued r.v. Then $X \in$ WLLN if and only if:*

i) $nP\{\|X\|>n\}\to 0$,

ii) $n^{-1}\sum_{\alpha}\delta_{n,\alpha}\to 0$,

iii) $\sum_{\alpha}\int_{\delta_{n,\alpha}}^{\infty} P\{|X_\alpha^n|>u\}\,du\to 0$,

iv) $n^{-\frac{1}{2}}\sum_{\alpha}\left[\int_{0}^{\delta_{n,\alpha}} uP\{|X_\alpha^n|>u\}\,du\right]^{\frac{1}{2}}\to 0$,

v) $\lim_n \sum_{\alpha} EX_\alpha^n e_\alpha$ *exists,*

where X_α^n denotes $(X^n)_\alpha$ and $\delta_{n,\alpha}$ is as in Theorem 3.3 for X_α^n, i.e.

$\delta_{n,\alpha}=\inf[t: nP\{|X_\alpha^n|>t\}\leq 1/8\cdot 3^p]$.

Proof. As observed in 5.1, by Lemma 2.6, $X \in$ WLLN if and only if $\varepsilon X \in$ WLLN and condition (v) holds. The quantities in (i)–(iv) above are the same for X as for εX. Hence we may and do assume that X is symmetric. Now the theorem follows directly from Lemma 5.2, Corollary 3.4 applied to $E\left|\sum_{i=1}^{n} X_{\alpha,i}\right|$, $\alpha=1,\ldots$, and from Lemma 3.2. □

5.3. Example. Let $X=\varepsilon \sum_{N^2\leq\alpha<N^2+N} e_\alpha \in l_1$, where ε is Rademacher, N is integer valued (and >0), and ε and N are independent. Then $\|X\|=N$ and $X_\alpha^n = \varepsilon I_{[N^2\leq\alpha<N^2+N, N\leq n]}$, so that $X_\alpha^n=0$ unless $k^2\leq\alpha<k^2+k$, $k\leq n$, and for such α $\delta_{n,\alpha}=1$ if $P\{N=k\}>1/24n$ and $\delta_{n,\alpha}=0$ if $P\{N=k\}\leq 1/24n$. So, the conditions in Theorem 5.3 read:

$$nP\{N>n\}\to 0, \qquad \sum_{k\leqq n,\, P\{N=k\}\leqq 1/24n} kP\{N=k\}\to 0,$$

$$n^{-\frac{1}{2}} \sum_{k\leqq n,\, P\{N=k\}>1/24n} k(P\{N=k\})^{\frac{1}{2}}\to 0.$$

For example, if $P\{N=k\}=c/k^2\ln k$ then the second of these conditions is violated (whereas the other two hold). If $P\{N=k\}=c/k^2(\ln k)(\ln\ln k)^\delta$, $\delta>0$, the three conditions hold and $X\in$WLLN.

5ii. WLLN in $l_1(l_p)$, $p>1$, and CLT in $l_2(l_p)$, $p>2$

The CLT in $l_2(l_p)$ will follow from the WLLN in $l_1(l_{p/2})$ by Theorem 2.9. For the WLLN in $l_1(l_p)$ we consider two cases, $p>2$ and $1\leqq p<2$, because in the first case we can use Rosenthal's inequalities and obtain a somewhat simpler result. If $X\in l_r(l_p)$, $r=1,2$, $p\geqq 1$, then $X=(X_\alpha)_{\alpha=1}^\infty$ with $X_\alpha\in l_p$, and we write $X_{\alpha\beta}$ for the β-th coordinate of X_α,

$$X_\alpha^n=X_\alpha I_{[\|X\|\leqq n]}, \qquad X_{\alpha\beta}^n=X_{\alpha\beta} I_{[\|X\|\leqq n]},$$

$$\delta_{n,\alpha}=\inf[t: P\{\|X_\alpha^n\|>t\}\leqq 1/24n],$$

$$\delta_{n,\alpha\beta}=\inf[t: P\{\|X_{\alpha\beta}^n\|>t\}\leqq 1/24n],$$

and $\{X_{\alpha i}^n\}$, $\{X_{\alpha\beta i}^n\}$ denote independent copies of $\{X_\alpha^n\}$ and $\{X_{\alpha\beta}^n\}$ respectively.

5.4. Theorem. *Let X be a $l_1(l_p)$-valued r.v., $p\geqq 2$. Then $X\in$WLLN if and only if:*

(i) $nP\{\|X\|>n\}\to\infty$,

(ii) $n^{-1}\sum_\alpha E\max_{i\leqq n}\|X_{\alpha i}^n\|\to 0$,

(iii) $n^{1/p-1}\sum_\alpha (E\|X_\alpha^n I_{[\|X_\alpha^n\|\leqq\delta_{n,\alpha}]}\|^p)^{1/p}\to 0$,

(iv) $n^{-\frac{1}{2}}\sum_\alpha [\sum_\beta \{E(X_{\alpha\beta}^n I_{[\|X_\alpha^n\|\leqq\delta_{n,\alpha}]})^2\}^{p/2}]^{1/p}\to 0$,

(v) $\lim_n EXI_{[\|X\|\leqq n]}$ converges.

Proof. Since conditions (i)–(iv) are identical for X and for εX, by Lemma 2.6 and condition (v) we may assume that X is symmetric (as in previous proofs). So, it is enough to check the conditions of Lemma 5.2. 5.2(i) = 5.4(i); as for the rest we first note that by Theorem 3.3,

$$n^{-1}E\left\|\sum_{i=1}^n X_{\alpha i}^n\right\|\simeq n^{-1}E\max_{i\leqq n}\|X_{\alpha i}^n\| + n^{-1}\left(E\left\|\sum_{i=1}^n X_{\alpha i}^n I_{[\|X_{\alpha i}^n\|\leqq\delta_{n,\alpha}]}\right\|^p\right)^{1/p} \tag{5.1}$$

where $\simeq$ means that each side is bounded above and below by some constant times the other side. Now, by Rosenthal's inequalities ([32], [34]),

$$\left(E\left\|\sum_{i=1}^{n} X_{\alpha i}^{n} I_{[\|X_{\alpha i}^{n}\| \leqq \delta_{n,\alpha}]}\right\|^{p}\right)^{1/p}$$

$$\simeq n^{1/p}(E\|X_{\alpha}^{n} I_{[\|X_{\alpha}^{n}\| \leqq \delta_{n,\alpha}]}\|^{p})^{1/p} + n^{1/2}[\sum_{\beta}\{E(X_{\alpha\beta}^{n} I_{[\|X_{\alpha}^{n}\| \leqq \delta_{n,\alpha}]})^{2}\}^{p/2}]^{1/p}. \quad (5.2)$$

Now the theorem follows from (5.1) and (5.2). □

Note that for $p=2$ conditions (iii) and (iv) in Theorem 5.4 coincide.

With the same proof, but using Theorem 3.3 in (5.2) instead of Rosenthal's inequality, we obtain:

5.5. Theorem. *Let X be a $l_1(l_p)$-valued r.v., $1<p<2$. Then $X \in$ WLLN if and only if:*

(i) $nP\{\|X\|>n\} \to 0$,

(ii) $n^{-1}\sum_{\alpha} E\max_{i\leqq n}\|X_{\alpha i}^{n}\| \to 0$,

(iii) $n^{-1}\sum_{\alpha}(\sum_{\beta} E\max_{i\leqq n}|X_{\alpha\beta i}^{n}|^{p} I_{[\|X_{\alpha i}^{n}\| \leqq \delta_{n,\alpha}]})^{1/p} \to 0$,

(iv) $n^{-\frac{1}{2}}\sum_{\alpha}(\sum_{\beta}[E|X_{\alpha\beta}^{n}|^{2} I_{[\|X_{\alpha}^{n}\| \leqq \delta_{n,\alpha}, |X_{\alpha\beta}^{n}| \leqq \delta_{n,\alpha\beta}]}]^{p/2})^{1/p} \to 0$,

(v) $\lim_{n} EXI_{[\|X\| \leqq n]}$ exists.

The interest of these last two theorems lies, in our view, in the fact that they give new necessary and sufficient conditions for the CLT in some of the Banach spaces where Theorem 4.3 does not apply. In fact we have:

5.6. Theorem. *Let Y be a centered $l_2(l_p)$-valued r.v., $p>2$. Then $Y \in$ CLT if and only if the $l_1(l_{p/2})$-valued r.v. $X=|Y|^2$ satisfies conditions* (i)–(v) *in* 5.4 *or* 5.5 *according as $p \geqq 4$ or $p<4$.*

Proof. Immediate from Theorems 2.9, 5.4 and 5.5 (note that $l_2(l_p)$, $p \geqq 2$, is of type 2). □

Remark. We do not know if there is any overlap in conditions (i)–(iv) in the previous theorems, although we suspect that there is not in general (possibly, in different particular cases there are different overlaps). Condition (ii) can not be dispensed with: for instance in Example 5.3 with $P\{N=k\}=c/k^2 \ln k$, the truncated moments condition is not violated and yet $X \notin$ WLLN.

6. Domains of Attraction in l_p, p<2

Theorem 3.3 can also be applied to the solution of the domains of attraction problem in l_p, $1 \leqq p<2$, and even the general central limit problem in l_p, $1 \leqq p<2$, at least formally (the conditions on $\mathscr{L}(X)$ may not always be easy to verify). In this section we will give a sample of what may be done, proving some general statements and testing them with examples. The case $0<p<1$ will also be briefly considered.

In what follows little attention will be paid to the norming and centering constants in connection with domains of attraction as it is well known what they should be in all cases (see e.g. [5], comments after 3.6.7).

We start with a useful symmetrization lemma.

6.1. Lemma. *Let X be a centered B-valued r.v., B a Banach space. Then $X \in DA_r(a_n)$, $r \in (1,2]$, if and only if:*

(i) $\varepsilon X \in \mathrm{DA}_r(a_n)$,

(ii) $f(X) \in \mathrm{DA}_r(a_n)$ *for every f in a sequentially w*-dense subset of B'.*

Proof. Let $b_n = n a_n^{-1} EXI_{[\|X\| \leqq a_n]}$. Then, if $X \in \mathrm{DA}_r(a_n)$, obviously (ii) holds and also $f(\varepsilon X) = \varepsilon f(X) \in \mathrm{DA}_r(a_n)$ for every $f \in B'$; by Lemma 2.6 we also have that $\left\{\mathscr{L}\left[\sum_{i=1}^{n} \varepsilon_i X_i/a_n - \left(\sum_{i=1}^{n} \varepsilon_i/n\right) b_n\right]\right\}$ is a tight sequence. Since $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/a_n - b_n\right)\right\}$ converges, say to ρ, then by Theorem 4.2 in [3] it follows that $nEX/a_n - b_n = -b_n \to \int x\,d\rho$. This implies by the law of large numbers that $\left\{\mathscr{L}\left(\sum_{i=1}^{n} \varepsilon_i X_i/a_n\right)\right\}$ is tight, hence convergent. Let us now assume that (i) and (ii) hold. By 2.6, (i) implies that $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/a_n - b_n\right)\right\}$ is tight, and another application (to subsequences) of 4.2 in [3] shows that $\{b_n\}$ is a relatively compact sequence in B; hence $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/a_n\right)\right\}$ is tight, and (ii) shows that it is actually convergent. □

Many of the cases we will consider can be based on the following lemma:

6.2. Lemma. *Let $X = \sum_{\alpha} X_\alpha e_\alpha$ be a centered l_p-valued r.v. $(1 \leqq p < 2)$ such that $f(X) \in \mathrm{DA}_r(a_n)$ for some (fixed) $r \in (p, 2]$ and for all $f \in D_p$. Then $X \in \mathrm{DA}_r(a_n)$ if and only if*

$$\lim_m \overline{\lim_n} \sum_{\alpha=m}^{\infty} E\left|\sum_{i=1}^{n} X_{\alpha i}/a_n\right|^p = 0 \tag{6.1}$$

and in fact $\overline{\lim_n}$ can be replaced by $\lim_n$.

Proof. If $X \in \mathrm{DA}_r(a_n)$, the centering constants may be taken to be 0 because X is centered, and Theorem 6.1 in [3] ([5], 3.6.18) implies that

$$\lim_m \lim_n \sum_{\alpha=m}^{\infty} E\left|\sum_{i=1}^{n} X_{\alpha i}/a_n\right|^p = \lim_m \int\left(\sum_{\alpha=m}^{\infty} |x_\alpha|^p\right) d\rho(x) = 0$$

$\left(\text{if } \rho \text{ is the limit law of } \sum_{i=1}^{n} X_i/a_n\right)$. Hence, condition (6.1) is necessary even with $\overline{\lim_n}$ replaced by $\lim_n$. If (6.1) holds then the sequence of laws $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/a_n\right)\right\}_{n=1}^{\infty}$ is flatly concentrated and therefore, by the condition on $f(X)$, $f \in D_p$, $X \in \mathrm{DA}_r(a_n)$ ([1], Theorem 2.3; see also [5], 1.4.5). □

The following theorem gives a solution to the domains of attraction problem for $1 \leq p < r \leq 2$.

6.3. Theorem. *Let* $X = \sum_\alpha X_\alpha e_\alpha$ *be a centered* l_p*-valued r.v.* $(1 \leq p < 2)$ *such that* $f(X) \in \mathrm{DA}_r(a_n)$ *for some (fixed)* $r \in (p, 2]$ *and for all* $f \in D_p$. *Then* $X \in \mathrm{DA}_r(a_n)$ *(in* l_p*) if and only if the following conditions hold:*

$$\lim_m \overline{\lim_n}\, a_n^{-p} \sum_{\alpha=m}^{\infty} \delta_{n,\alpha}^p = 0, \tag{6.2}$$

$$\lim_m \overline{\lim_n}\, n a_n^{-p} \sum_{\alpha=m}^{\infty} \int_{\delta_{n,\alpha}}^{\infty} t^{p-1} P\{|X_\alpha| > t\}\, dt = 0, \tag{6.3}$$

$$\lim_m \overline{\lim_n}\, n^{p/2} a_n^{-p} \sum_{\alpha=m}^{\infty} \left(\int_0^{\delta_{n,\alpha}} t P\{|X_\alpha| > t\}\, dt \right)^{p/2} = 0, \tag{6.4}$$

where $\delta_{n,\alpha} = \inf[t : nP\{|X_\alpha| > t\} \leq 1/8 \cdot 3^p]$, $\alpha = 1, \ldots$. *If in* (6.2)–(6.4) $\delta_{n,\alpha}$ *is replaced by* $\delta'_{n,\alpha} \geq \delta_{n,\alpha}$, *the resulting conditions are also sufficient for* $X \in \mathrm{DA}_r(a_n)$.

Proof. By Lemma 3.1 it is enough to consider symmetric variables. Then, Corollary 3.4 (together with Lemma 3.2) applied to $\sum_{i=1}^{n} X_{\alpha i}/a_n$, $\alpha \in \mathbb{N}$, together with Lemma 6.2, give the necessity and sufficiency of (6.2)–(6.4). That (6.2)–(6.4) with $\delta_{n,\alpha}$ replaced by $\delta'_{n,\alpha}$ are also sufficient for $X \in \mathrm{DA}_r(a_n)$, follows trivially from the observation that the right side inequality in (3.2) holds with δ_0 replaced by any $\delta'_0 \geq \delta_0$, as the proof of Lemma 3.2 shows. □

Here there are some examples and corollaries.

6.4. Example (well known). Let X be a centered l_p-valued r.v., $1 \leq p \leq 2$. Then $X \in \mathrm{CLT}$ if and only if

$$\sum_\alpha (E|X_\alpha|^2)^{p/2} < \infty. \tag{6.5}$$

Proof. By Čebyšev's inequality, $\delta_{n,\alpha} \leq \delta'_{n,\alpha} = 4 \cdot 3^{p/2} n^{\frac{1}{2}} (E|X_\alpha|^2)^{\frac{1}{2}}$. Then (6.2)–(6.4) for $\delta'_{n,\alpha}$ and $a_n = n^{\frac{1}{2}}$ follow trivially from (6.5), and the sufficiency is proved. To prove necessity we just observe that

$$\begin{aligned} &\underline{\lim_n} \sum_{\alpha=m}^{\infty} \left(\int_0^{\delta_{n,\alpha}} t P\{|X_\alpha| > t\}\, dt \right)^{p/2} \\ &\quad \geq \sum_{\alpha=m}^{\infty} \lim_n \left(\int_0^{\delta_{n,\alpha}} t P\{|X_\alpha| > t\}\, dt \right)^{p/2} = \sum_{\alpha=m}^{\infty} (E|X_\alpha|^2)^{p/2}, \end{aligned}$$

so that (6.5) follows from (6.4) (with $a_n = n^{\frac{1}{2}}$). □

6.5. Example. Let η be a centered real r.v. such that

$$t^2 P\{|\eta| > t\} / E|\eta|^2 I_{[|\eta| \leq t]} \to 1 \quad \text{as } t \to \infty,$$

and let $\{\eta_\alpha\}_{\alpha=1}^{\infty}$ be independent copies of η. Then the series $X=\sum_\alpha b_\alpha \eta_\alpha e_\alpha$ converges a.s. in l_p $(1\leq p<2)$ and belongs to the domain of attraction of a Gaussian law γ on l_p if and only if

$$\sum_\alpha |b_\alpha|^p<\infty.$$

In that case $\gamma=\mathscr{L}(\sum_\alpha b_\alpha \psi_\alpha e_\alpha)$, ψ_α i.i.d. $N(0,1)$, a_n is defined by $n a_n^{-2} E\eta^2 I_{[|\eta|\leq a_n]}\simeq 1$ and $b_n=0$.

Proof. Let $\delta_n=\inf[t: P\{|\eta|>t\}\leq 1/8\cdot 3^p n]$. Then the expressions (6.2)–(6.4) become respectively

$$a_n^{-p}\delta_n^{-p}\sum_{\alpha=m}^{\infty}|b_\alpha|^p, \qquad n a_n^{-p}\left(\int_{\delta_n}^{\infty} t^{p-1} P\{|\eta|>t\}\,dt\right)\sum_{\alpha=m}^{\infty}|b_\alpha|^p,$$

$$n^{p/2} a_n^{-p}(E\eta^2 I_{[|\eta|\leq \delta_n]})^{p/2}\sum_{\alpha=m}^{\infty}|b_\alpha|^p.$$

If we apply to η the fact that $\left\{E\left|\sum_{i=1}^{n}\eta_i/a_n\right|^p\right\}_{n=1}^{\infty}$ converges and Corollary 3.4, we conclude that the $\lim_m \overline{\lim_n}$ of the above expressions is zero if and only if $\sum_\alpha |b_\alpha|^p<\infty$. It is trivial that $f(\sum_\alpha b_\alpha \eta_\alpha e_\alpha)$, $f\in D_p$, is in the $\mathrm{DA}_r(a_n)$ of $\gamma\circ f^{-1}$. So, Theorem 6.3 gives the result. □

Theorem 6.3, applied to domains of normal attraction of r-stable laws, $p<r<2$, gives the following simple result:

6.6. Proposition. *Let $1\leq p<r<2$. Let $X=\sum_\alpha X_\alpha e_\alpha$ be a centered l_p-valued r.v. such that $f(X)$ is in the domain of normal attraction of a stable law in $\mathbb{R}$ for every $f\in D_p$. Then,*

$$\sum_\alpha [\sup_{t>0} t^r P\{|X_\alpha|>t\}]^{p/r}<\infty \Rightarrow X\in \mathrm{DNA}_r, \tag{6.6}$$

$$X\in \mathrm{DNA}_r \Rightarrow \sum_\alpha [\lim_{t\to\infty} t^r P\{|X_\alpha|>t\}]^{p/r}<\infty. \tag{6.7}$$

Proof. Set $\Lambda_{r,\alpha}=[\sup_{t>0} t^r P\{|X_\alpha|>t\}]^{1/r}$ and $\delta'_{n,\alpha}=8^{1/r}\cdot 3^{p/r}\Lambda_{r,\alpha} n^{1/r}$. Then obviously $\delta_{n,\alpha}\leq\delta'_{n,\alpha}$, and we have

$$n^{-p/r}\sum_{\alpha=m}^{\infty}(\Lambda'_{n,\alpha})^p=8^{p/r}\cdot 3^{p^2/r}\cdot\sum_{\alpha=m}^{\infty}(\Lambda_{r,\alpha})^p,$$

$$n^{1-p/r}\sum_{\alpha=m}^{\infty}\int_{\delta'_{n,\alpha}}^{\infty} t^{p-1} P\{|X_\alpha|>t\}\,dt$$

$$\leq 8^{p/r-1}\cdot 3^{p(p/r-1)}(r-p)^{-1}\sum_{\alpha=m}^{\infty}(\Lambda_{r,\alpha})^p,$$

$$n^{p/2-p/r}\sum_{\alpha=m}^{\infty}\left(\int_0^{\delta'_{n,\alpha}} tP\{|X_\alpha|>t\}\,dt\right)^{p/2}$$
$$\leqq 8^{p(2-r)/2r}\cdot 3^{p^2(2-r)/2r}(2-r)^{-p/2}\sum_{\alpha=m}^{\infty}(\Lambda_{r,\alpha})^p.$$

Therefore, $\sum_{\alpha=m}^{\infty}(\Lambda_{r,\alpha})^p<\infty$ implies (6.2)-(6.4) for $\delta'_{n,\alpha}$ and $a_n=n^{1/r}$, hence that $X\in\mathrm{DNA}_r$ by Theorem 6.3. Let now $\lambda_{r,\alpha}=\lim_{t\to\infty}[t^r P\{|X_\alpha|>t\}]^{1/r}$. Then for each $\alpha>0$ there exists $t_\alpha<\infty$ such that for all $t>t_\alpha$,

$$t^r P\{|X_\alpha|>t\}\geqq(\lambda_{r,\alpha})^r/2$$

so that for n large enough (depending on α), $P\{|X_\alpha|>n^{1/r}\lambda_{r,\alpha}\}\geqq 1/2n$. Hence $\delta_{n,\alpha}\geqq n^{1/r}(\lambda_{r,\alpha})^{1/r}$. This, together with the necessity of (6.2), gives statement (6.7). □

Remark. If, in the situation of Proposition 6.6, there exist c_α, $\alpha=1,\ldots$, such that $\lim_{t\to\infty} t^r P\{|X_\alpha|/c_\alpha\}>t\}=c\in(0,\infty)$ uniformly in α, then the necessary and the sufficient conditions in Proposition 6.6 coincide. But this is not true in general, as we will show next.

Before proceeding with domains of attraction in other cases, some comments concerning 6.6 are in order. Two questions come to mind which should be answered: one is whether the condition in (6.7) is also sufficient for $X\in\mathrm{DNA}_r$, and the other, whether is it true in l_p, $p<r$, that $X\in\mathrm{DNA}_r$ if and only if X is *r-pre-stable*, as in the case $r=2$. Here X r-pre-stable means that there exists a r-stable r.v. Y in l_p such that $f(X)\in\mathrm{DNA}_r(f(Y))$ for every $f\in D_p$. The answer to both questions is negative and will be given by the same example. But before we do that we must obtain the precise form of all symmetric r-stable laws in l_p, $2>r>p\geqq 1$. In the next proposition e^*_α is the linear form conjugate to e_α.

6.7. Proposition. *Let ρ be a symmetric r-stable cylindrical p.m. on l_p, $2>r>p\geqq 1$. Then ρ is tight if and only if*

$$\sum_\alpha[\lim_{t\to\infty} t^r\rho\{x:|e^*_\alpha(x)|>t\}]^{p/r}<\infty. \tag{6.8}$$

Equivalently, let Y_α be symmetric r-stable real r.v.'s such that for all finite sets $I\subset\mathbb{N}$ and $\beta_\alpha\in\mathbb{R}$, $\sum_{\alpha\in I}\beta_\alpha Y_\alpha$ is r-stable; then the series $\sum_\alpha Y_\alpha e_\alpha$ converges in law (in probability, in L_p) in l_p if and only if

$$\sum_\alpha[\lim_{t\to\infty} t^r P\{|Y_\alpha|>t\}]^{p/r}<\infty. \tag{6.9}$$

Equivalently, let σ be a finite measure on $S=\{x\in l_p:\|x\|=1\}$. Then the measure $d\sigma(s)\,du/u^{p+1}$ on $l_p-\{0\}$ is the Lévy measure of a r-stable law on l_p if and only if

$$\sum_\alpha[\int_S|\langle e^*_\alpha,s\rangle|^r\,d\sigma(s)]^{p/r}<\infty. \tag{6.10}$$

Proof. We will prove the second formulation, which is obviously equivalent to the first and it is easily seen to be equivalent to the third using the representation theorem for stable laws in Banach spaces – see e.g. 3.6.16 in [5]. If (6.9) holds, then

$$E\left\|\sum_{\alpha=k}^{m} Y_\alpha e_\alpha\right\|^p = \sum_{\alpha=k}^{m} E|Y_\alpha|^p \leqq c \sum_{\alpha=k}^{m} [\lim_{t\to\infty} t^r P\{|Y_\alpha|>t\}]^{p/r}$$

for some $c<\infty$. Hence the series $\sum_\alpha Y_\alpha e_\alpha$ is Cauchy in L_p and therefore it converges in L_p and in probability, and in law). Conversely, if the series $\sum_\alpha Y_\alpha e_\alpha$ converges in law to ρ, then the limit ρ is r-stable and therefore it integrates the $p(1+\varepsilon)$-th power of the norm for any ε such that $p(1+\varepsilon)<r$. Hence, the sequence $\left\{\mathscr{L}\left(\sum_{\alpha=1}^{m}|Y_\alpha|^p\right)\right\}_{m=1}^{\infty}$ converges weakly to a p.m. on $\mathbb{R}$ with $(1+\varepsilon)$-th absolute moment finite; then, since these r.v.'s increase, it follows that $\sum_{\alpha=1}^{\infty} E|Y_\alpha|^p<\infty$. Since

$$E|Y_\alpha|^p \geqq c'[\lim_{t\to\infty} t^r P\{|Y_\alpha|>t\}]^{p/r},$$

the necessity of (6.9) is proved. □

Remark. In its third version, this lemma is due to A. Račkauskas [30]. We have not been able to see his article; in any case the previous proof is very simple.

By Proposition 6.7 a random variable $X=\sum_\alpha X_\alpha e_\alpha \in l_p$, with the X_α independent and symmetric, is r-pre-stable if and only if

$$\sum_\alpha [\lim_{t\to\infty} t^r P\{|X_\alpha|>t\}]^{p/r}<\infty. \tag{6.11}$$

However, it is possible to produce an example of such an X such that $E\|X\|^p=\infty$, which means in particular that $X\notin \mathrm{DNA}_r$ and that X does not satisfy condition (6.6) (see e.g. Theorem 3.6.18(2) in [5]). Take

$$X_\alpha=\tau_\alpha a_\alpha \theta_\alpha+(1-\tau_\alpha) w_\alpha \varepsilon_\alpha, \alpha=1,\ldots, \quad \text{and} \quad X=\sum_\alpha X_\alpha e_\alpha,$$

where $\{\tau_\alpha, \theta_\alpha, \varepsilon_\alpha\}_{\alpha=1}^{\infty}$ are all independent, θ_α are i.i.d. symmetric stable of order r, $\{\varepsilon_\alpha\}$ is a Rademacher sequence, τ_α is 0 with probability $1/\alpha^2$ and 1 with probability $1-1/\alpha^2$, $w_\alpha=\alpha^{1/p}$ and $\sum_\alpha |a_\alpha|^p<\infty$. Then, $\sum_\alpha \tau_\alpha a_\alpha \theta_\alpha e_\alpha$ converges by Proposition 6.7 and Ito-Nisio's Lemma, and $\sum_\alpha (1-\tau_\alpha) w_\alpha \varepsilon_\alpha e_\alpha$ converges a.s. by the Borel-Cantelli lemma. Hence X is well defined as a l_p-valued r.v. For $t>w_\alpha$

$$t^r P\{|X_\alpha|>t\}=(1-\alpha^{-2}) t^r P\{|a_\alpha \theta_\alpha|>t\}$$

so that

$$\lim_{t\to\infty} t^r P\{|X_\alpha|>t\}=c(1-\alpha^{-2})|a_\alpha|^r$$

and X is r-pre-stable. On the other hand,

$$E\|X\|^p=\sum_\alpha(1-\alpha^{-2})|a_\alpha|^p E|\theta|^p+\sum_\alpha \alpha^{-1}=\infty.$$

Summarizing:

6.8. Proposition. *For $2>r>p\geqq 1$, there exist l_p-valued r.v.'s X such that all their finite dimensional distributions belong to the DNA of the corresponding finite dimensional distributions of a r-stable law in l_p, but $X\notin \mathrm{DNA}_r$. Such X necessarily satisfy* (6.11).

The case $p>r$ in the domains of attraction problem is completely solved (as l_p is of type $\min(2,p)$); see e.g. [4, 24, 33] (also [9, 5]). Just for comparison with Theorem 6.3 and for use in some examples below, let us record that if the finite dimensional distributions of $X\in l_p$ are in $\mathrm{DA}_r(a_n)$'s, $r<\min(2,p)$, then $X\in\mathrm{DA}_r(a_n)$ if and only if

$$\lim_m \overline{\lim_n}\, nP\left\{\left\|\sum_{\alpha=m}^{\infty} X_\alpha e_\alpha\right\|>a_n\right\}=0. \tag{6.12}$$

Next we consider the case $1\leqq p=r<2$. Given $X=\sum_\alpha X_\alpha e_\alpha\in l_p$, let us set $X_\alpha^n=X_\alpha I_{[\|X\|\leqq a_n]}$ and let $\delta_{n,\alpha}$ refer to X_α^n (and not to X_α as in Theorem 6.3). In case $p=1$ we let $Y_\alpha^n=X_\alpha^n-EX_\alpha^n$ and let $\delta_{n,\alpha}$ refer to Y_α^n. Define also $Y^n=\sum_\alpha Y_\alpha^n e_\alpha$. Then we have:

6.9. Theorem. *Let $1<p<2$ and let X be a centered l_p-valued r.v. such that $f(X)\in\mathrm{DA}_p(a_n)$ for all $f\in D_p$. Then $X\in\mathrm{DA}_p(a_n)$ (in l_p) if and only if the following conditions hold:*

$$\lim_m \overline{\lim_n}\, nP\left\{\left\|\sum_{\alpha=m}^{\infty} X_\alpha e_\alpha\right\|>a_n\right\}=0, \tag{6.13}$$

$$\lim_m \overline{\lim_n}\, a_n^{-p}\sum_{\alpha=m}^{\infty}\delta_{n,\alpha}^p=0, \tag{6.14}$$

$$\lim_m \overline{\lim_n}\, na_n^{-p}\sum_{\alpha=m}^{\infty}\int_{\delta_{n,\alpha}}^{\infty} t^{p-1}P\{|X_\alpha^n|>t\}\,dt=0, \tag{6.15}$$

$$\lim_m \overline{\lim_n}\, n^{p/2}a_n^{-p}\sum_{\alpha=m}^{\infty}\left(\int_0^{\delta_{n,\alpha}} tP\{|X_\alpha|>t\}\,dt\right)^{p/2}=0. \tag{6.16}$$

If $p=1$ the same result holds for any X (X cannot be centered) with X_α^n replaced by Y_α^n in (6.14)–(6.16). The conditions (6.14)–(6.16) for $\delta'_{n,\alpha}\geqq\delta_{n,\alpha}$ are also sufficient for $X\in\mathrm{DA}_p(a_n)$.

Proof. Case $p\neq 1$. X may be assumed to be symmetric by Lemma 6.1. With the above assumptions on $f(X)$ (X symmetric), we have that $X\in\mathrm{DA}_p(a_n)$ if and only if (6.13) holds and

$$\lim_m \overline{\lim_n}\, a_n^{-p}E\left\|\sum_{\alpha=m}^{\infty}\left(\sum_{i=1}^{n}X_{\alpha i}^n\right)e_\alpha\right\|^p=0. \tag{6.17}$$

(This follows e.g. from Proposition 3.1 in [9] and Theorem 3.5.9 in [5] - 2.10, 2.14 in [2]). The first part of the proposition follows from this, Corollary 3.4 and Lemma 3.2.

Case $p=1$. $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i I_{[\|X_i\|>a_n]}/a_n\right)\right\}_{n=1}^{\infty}$ is tight if (6.13) holds (3.1 in [9] together with 2.1 and 2.4 in [2]). By Lemma 2.6, $\left\{\mathscr{L}\left(\sum_{i=1}^{n} Y_i^n/a_n\right)\right\}_{n=1}^{\infty}$ is tight if and only if $\left\{\mathscr{L}\left(\sum_{i=1}^{n} \varepsilon_i Y_i^n/a_n\right)\right\}_{n=1}^{\infty}$ is tight, so that the Y_α^n can be assumed to be symmetric. (6.19) for Y_α^n is necessary and sufficient for the tightness of $\left\{\mathscr{L}\left(\sum_{i=1}^{n} Y_i^n/a_n\right)\right\}_{n=1}^{\infty}$. Now the result is obtained as before by application of 3.2 and 3.4. Finally, the case $\delta'_{n,\alpha}\geqq\delta_{n,\alpha}$ follows as in Theorem 6.3. □

Remark (Domains of attraction in l_p, $0<p<1$). It is not difficult to prove that Theorems 6.3 and 6.9, as well as the necessity and sufficiency of condition (6.12) (together with the conditions on $f(X)$) for $X\in \mathrm{DA}_r(a_n)$ are also true for symmetric l_p-valued r.v.'s, $0<p<1$. (The case $p>r$ for series of independent summands was already observed in [24]). We will sketch the proof of Theorem 6.3 and leave the rest to the reader. Since Corollary 3.4 holds for $p<1$, it is enough to prove Lemma 6.2. Let X_i be independent, symmetric l_p-valued r.v.'s and $S_k=\sum_{i=1}^{k} X_i$. Then it can be proved just as in the Banach space case that the following version of the Lévy inequalities holds: for all $t>0$ and $n=1,\dots,$

$$P\{\max_{k\leqq n}\|S_k\|_p>t\}\leqq 2P\{\|S_n\|_p>2^{p-1}t\},$$

$$P\{\max_{i\leqq n}\|X_i\|_p>t\}\leqq 2P\{\|S_n\|_p>2^{p-1}t\},$$

where, if $x=\sum_\alpha x_\alpha e_\alpha\in l_p$, then $\|x\|_p=\sum_\alpha |x_\alpha|^p$. This implies that Theorem 6.1 in [3] on convergence of moments in the domains of attraction limit theorems is true in l_p (with $\|\cdot\|$ replaced by $\|\cdot\|_p^{1/p}$). This gives the necessity of (6.1). For the sufficiency let us first observe that a set $K\subset l_p$ is relatively compact if and only if

$$\limsup_m \left\{\sum_{\alpha=m}^{\infty}|x_\alpha|^p : x\in K\right\}=0,$$

with the sup finite for $m=1$ (necessity follows from Dini's lemma, and sufficiency from total boundedness). This implies that if $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/a_n\right)\right\}_{n=1}^{\infty}$ satisfies (6.1), then it is tight. The result is thus proved. The observations on Lévy's inequalities and on compact sets are also useful in proving the remaining cases (Theorem 6.8 and condition (6.12)). Note for instance that the tightness of $\left\{\mathscr{L}\left(\sum_{i=1}^{n} X_i/a_n\right)\right\}_{n=1}^{\infty}$ readily implies by Lévy's inequality and the above characterization of compact sets that

$$\limsup_m \sup_n nP\left\{\left\|\sum_{\alpha=m}^{\infty} X_\alpha e_\alpha\right\|_p > a_n^p\right\}$$
$$\leq \limsup_m \sup_n \left[-\ln\left(1-2P\left\{\left\|\sum_{\alpha=m}^{\infty}\sum_{i=1}^{n} X_{\alpha i} e_\alpha\right\|_p > 2^{p-1} a_n^p\right\}\right)\right] = 0,$$

hence (6.12). The rest of the proofs follow, as this one, by relatively simple modifications of the corresponding arguments for Banach spaces.

Now we will give some straightforward applications of Proposition 6.6, Theorem 6.9 and, for comparison purposes, criterion (6.12) for spaces of type $p>r$. In what follows, a standard r-stable r.v. θ, $r<2$, is a r.v. with characteristic function $Ee^{it\theta}=e^{-c|t|^r}$, $-\infty<t<\infty$, with $c=2r\int_{-\infty}^{\infty}(1-\cos u)\,du/|u|^{1+r}$, so that $t^r P\{|\theta|>t\}\to 1$ as $t\to\infty$. These applications are easy consequences of the previous propositions and some well known facts on stable r.v.'s in $\mathbb{R}$. So, only the first example, case $1\leq p<r$, will be at all discussed.

6.10. Example. Consider the series $\sum_i \eta_i x_i$, $x_i\in l_p$, $0<p<2$, η_i independent, symmetric, such that

$$\lim_{t\to\infty} t^r P\{|\eta_i|>t\}=1$$

uniformly in i for some $0<r<2$. Set $x_i=\sum_\alpha x_i^\alpha e_\alpha$. Then a necessary and sufficient condition for $X=\sum_i \eta_i x_i$ to converge (a.s. in l_p) and to belong to the DNA$_r$ of $Y=\sum_i \theta_i x_i$, θ_i i.i.d. standard r-stable symmetric r.v.'s, is:

$$\sum_\alpha\Big(\sum_i |x_i^\alpha|^r\Big)^{p/r}<\infty \qquad \text{if } p<r, \tag{6.18}$$

and

$$\sum_i \|x_i\|^r<\infty \quad \text{if } p>r \qquad \Big(\|x_i\|=\Big(\sum_\alpha |x_i^\alpha|^p\Big)^{1/p} \text{ even if } p<1\Big). \tag{6.19}$$

If $p=r$, a sufficient condition is

$$\sum_\alpha\Big(\sum_i |x_i^\alpha|^p\Big)\max\Big[1,\ln\sum_i |x_i^\alpha|^p|\Big]<\infty \tag{6.20}$$

(if $x_i=e_i$ this condition is also necessary; condition (6.18) for $p=r$ $(=(6.19))$ is always necessary, and is also sufficient if $x_i=\lambda_i x$, $\lambda_i\in\mathbb{R}$, $x\in l_p$, $i=1,\ldots$).

Proof of the Case $1\leq p<r$. It is well known that $\sum_i a_i\eta_i$, $a_i\in\mathbb{R}$, converges a.s. if and only if $\sum_i |a_i|^r<\infty$ (three series theorem) and that in this case

$$\sup_{t>0} t^r P\Big\{\Big|\sum_i a_i\eta_i\Big|>t\Big\}\leq(4-r)(2-r)^{-1}\sum_i |a_i|^r, \tag{6.21}$$

$$\lim_{t\to\infty} t^r P\Big\{\Big|\sum_i a_i\eta_i\Big|>t\Big\}=\sum_i |a_i|^r, \tag{6.22}$$

and likewise if η_i is replaced by θ_i (see e.g. Lemma 2.1 in [12] for (6.21); (6.22) is obtained from (6.21) by means of two elementary inequalities - see e.g. ex. 3.7.11 and 12 in [5]). (6.21) with $a_i = x_i^\alpha$, together with (6.18), imply that $\sum_\alpha X_\alpha e_\alpha = \sum_\alpha (\sum_i \eta_i x_i^\alpha) e_\alpha$ converges in probability, hence that $\sum_i \eta_i x_i$ converges a.s. by Ito-Nisio's lemma. If $f \in D_p$, then (6.22) with $a_i = f(x_i)$ shows that $f(X) \in \mathrm{DNA}_r$ of $f(Y)$. Finally, (6.21) for $a_i = x_i^\alpha$, together with (6.18), shows that

$$\sum_\alpha [\sup_{t>0} t^r P\{|X_\alpha| > t\}]^{p/r} < \infty.$$

Hence, by Proposition 6.6, condition (6.18) is sufficient for X to be in the DNA_r of Y. The necessity of (6.18) follows from the same proposition, (6.7), together with (6.22) applied to $a_i = x_i^\alpha$. □

Remark. Note that condition (6.18) is equivalent to X being pre-stable. This is in direct contrast with Proposition 6.8. This phenomenon is explained by the remark following Proposition 6.6, and it was already observed in [24], Theorem 6.1 and the comments following it. Part of the previous example can also be proved using Theorem 6.1 in [24]. This is not the case with the following example.

6.11. Example. Let η be a symmetric real r.v. in the DNA_r of θ, standard stable of order r, and let N be an integer valued r.v. independent of η. Then, for $0 < p < 2$, the l_p-valued r.v. $\eta \sum_{N^2 \leq \alpha < N^2+N} e_\alpha \in \mathrm{DNA}_r$ if and only if:

$$\sum_{k=1}^{\infty} k(P\{N=k\})^{p/r} < \infty \qquad \text{if } p < r,$$

$$\sum_{k=1}^{\infty} k^{r/p} P\{N=k\} < \infty \qquad \text{if } p > r,$$

$$\sum_{k=1}^{\infty} kP\{N=k\} \max[1, |\ln kP\{N=k\}|] < \infty \qquad \text{if } p = r.$$

Remark (the general CLT in l_p, $1 \leq p < 2$). Let us finally mention that it is possible to obtain a general CLT for l_p-valued $(1 \leq p < 2)$ row-wise independent arrays $\{X_{nj}\}$ in terms of the laws of the individual variables X_{nj}: in Theorems 2.10, 2.14 [2], apply Theorem 3.3 to the truncated sums after using the symmetrization Lemma 2.6. The possibility of doing it being apparently more interesting than the result itself, we choose not to state such a theorem. Similar comments apply, at least for symmetric variables, in the case $0 < p < 1$.

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References

1. de Acosta, A.: Existence and convergence of probability measures in Banach spaces. Trans. Amer. Math. Soc. **152**, 273–298 (1970)
2. de Acosta, A., Araujo, A., Giné, E.: On Poisson measures, Gaussian measures and the central limit theorem in Banach spaces. Adv. in Probability, Vol. IV (Probability in Banach spaces, J. Kuelbs, ed.) 1–68. New York: Dekker 1978
3. de Acosta, A., Giné, E.: Convergence of moments and related functionals in the central limit theorem in Banach spaces. Z. Wahrscheinlichkeitstheorie verw. Geb. **48**, 213–231 (1979)
4. Araujo, A., Giné, E.: On tails and domains of attraction of stable measures on Banach spaces. Trans. Amer. Math. Soc. **248**, 105–119 (1979)
5. Araujo, A., Giné, E.: The central limit theorem for real and Banach valued random variables. New York: Wiley 1980
6. Chobanian, S.A., Tarieladze, V.I.: A counterexample concerning CLT in Banach spaces. Lecture Notes in Math. **656**, 25–30. Berlin-Heidelberg-New York: Springer 1978
7. Denker, M., Kombrink, R.: On B-convex Orlicz spaces. Lecture Notes in Math. **709**, 87–96. Berlin-Heidelberg-New York: Springer 1979
8. Figiel, T.: On the moduli of convexity and smoothness. Studia Math. **56**, 121–155 (1976)
9. Giné, E.: Domains of attraction in Banach spaces. Lecture Notes in Math. **721**, 22–40. Berlin-Heidelberg-New York: Springer 1979
10. Giné, E.: Sums of independent random variables and sums of their squares. Publ. Mat. Univ. Autònoma de Barcelona **22**, 127–132 (1980)
11. Giné, E., Mandrekar, V., Zinn, J.: On sums of independent random variables with values in L_p, $p \geqq 2$. Lecture Notes in Math. **709**, 111–124. Berlin-Heidelberg-New York: Springer 1979
12. Giné, E., Marcus, M.B.: Some results on the domain of attraction of stable measures in $C(K)$. [Probability and Math. Statist. To appear]
13. Hoffmann-Jørgensen, J.: Sums of independent Banach space valued random variables. Studia Math. **52**, 159–186 (1974)
14. Hoffmann-Jørgensen, J., Pisier, G.: The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. **4**, 587–599 (1976)
15. Johnson, W.B.: Banach spaces all of whose subspaces have the approximation property. Special topics of Applied Mathematics (Frehse et al., ed.), 14–26. Amsterdam: North Holland 1980
16. Jain, N.: Central limit theorem and related questions in Banach spaces. Proc. Symp. in Pure Math. XXXI, 55–65. Amer. Math. Soc., Providence, R.I., 1977
17. Klass, M.: Precision bounds for the relative error in the approximation of $E|S_n|$ and extensions. Ann. Probab. **8**, 350–367 (1980)
18. Krasnoselskii, M.A., Rutikii, Y.B.: Convex functions and Orlicz spaces. Groningen 1961 [Transl. from Russian]
19. Krivine, J.L.: Théorèmes de factorisation dans les espaces reticulés. Séminaire Maurey-Schwartz 1973–74. Exp. XXII et XXIII. Ecole Polytechnique, Paris
20. Kuelbs, J., Zinn, J.: Some results on LIL behavior. To appear in Ann. Probab.
21. Lindenstrauss, J., Tzafriri, L.: Classical Banach spaces I. Berlin-Heidelberg-New York: Springer 1977
22. Lindenstrauss, J., Tsafriri, L.: Classical Banach spaces II. Berlin-Heidelberg-New York: Springer 1979
23. Mandrekar, V., Zinn, J.: Central limit theorem for symmetric case: convergence to non-Gaussian laws. Studia Math. **67**, 279–296 (1980)
24. Marcus, M., Woyczynski, W.: Stable measures and central limit theorems in spaces of stable type. Trans. Amer. Math. Soc. **251**, 71–101 (1979)
25. Maurey, B.: Type et cotype dans les espaces minis de structures locales inconditionelles. Séminaire Maurey-Schwartz 1973–74. Exp. XXIV et XXV. Ecole Polytechnique, Paris
26. Maurey, B., Pisier, G.: Séries de variables aléatoires independentes et proprietés géométriques des espaces de Banach. Studia Math. **58**, 45–90 (1976)
27. Mourier, E.: Elements aléatoires à valeurs dans un espace de Banach. Ann. Inst. H. Poincaré **13**, 159–244 (1953)

28. Pisier, G.: Le théorème limite central et la loi du logaritme iteré dans les espaces de Banach. Séminaire Maurey-Schwartz 1975–76. Exp. III et IV. Ecole Polytechnique, Paris
29. Pisier, G., Zinn, J.: On the limit theorems for random variables with values in L_p, $p \geqq 2$. Z. Wahrscheinlichkeitstheorie verw. Gebiete **41**, 289–304 (1978)
30. Račkauskas, A.: A remark on stable measures on Banach spaces. Lithuanian Math. J. **19**, 267–270 (1979)
31. Reisner, S.: A factorization theorem in Banach lattices and its application to Lorentz spaces. Ann. Inst. Fourier (Grenoble) **31**, 239–255 (1981)
32. Rosenthal, H.: On the span in L_p of sequences of independent random variables. Proc. Sixth Berkeley Sympos. on Math. Statist. and Probab. Vol. II, 149–167. Univ. of California Press, Berkeley (1972)
33. Woyczynski, W.: Geometry and martingales in Banach spaces. Part II: independent increments. Adv. in Probability, Vol. 4 (Probability in Banach spaces, J. Kuelbs ed.) 267–518. New York: Dekker 1978
34. Zinn, J.: Inequalities in Banach spaces with applications to probabilistic limit theorems: a survey. Lecture Notes in Math. **860**, 324–329. Berlin-Heidelberg-New York: Springer 1981
35. Marcus, M.B., Pisier, G.: Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes (1982). Preprint.

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Addendum. To complement this work we give a characterization of symmetric p-stable laws in l_p, $0<p<2$. G. Pisier noted that Hoffmann-Jorgensen's lemma together with the lemma below gives this characterization. This lemma follows from Remark 3.15 in [35]. Let B be a separable Banach space and $S=\{x \in B: \|x\|=1\}$. We say that a finite measure σ on S is a spectral measure of a p-stable law μ, on B, $0<p<2$, if for all $f \in B$

$$\hat{\mu}(f)=\exp\{\int_{S \times \mathbb{R}^+} (e^{irf(s)}-1-irf(s)\, I_{[r \leqq 1]})\, d\sigma(s)\, dr/r^{1+p}\}.$$

6.12. Lemma. *Let σ be a finite measure on B. Let $\{X_i\}$ be i.i.d. S-valued r.v.'s $\mathcal{L}(X_i)=\sigma/\sigma(S)$, and $\{\varepsilon_i\}$ a Rademacher sequence independent of $\{X_i\}$. Then σ is the spectral measure of a p-stable law on B if and only if the series*

$$\sum_j \varepsilon_j X_j/j^{1/p} \tag{6.23}$$

is a.s. (or L_r, $0<r<\infty$), convergent.

6.13. Proposition. *Let σ be a finite measure on $S=\{x: \|x\|=1\} \subset l_p$, $1 \leqq p \leqq 2$, and let X be a l_p-valued r.v. such that $\mathcal{L}(X)=\sigma/\sigma(S)$. Then σ is the spectral measure of a p-stable law on l_p if and only if*

$$\sum_\alpha E|X_\alpha|^p \left(1+\ln_+ \frac{|X_\alpha|}{\|X_\alpha\|_p}\right)<\infty \tag{6.24}$$

where $X=\sum_\alpha X_\alpha e_\alpha$, $\|X_\alpha\|_p=(E|X_\alpha|^p)^{1/p}$ and $\ln_+(a)=\max(0, \ln a)$.

Proof (Sketch). We may and do assume X symmetric. By Theorem 3.3 the following conditions are necessary and sufficient for convergence of (6.23):

(a) $\sum_\alpha \delta^p < \infty$,

(b) $\sum_\alpha [\sum_j j^{-2/p} E X_\alpha^2 I_{[|X_\alpha| \leqq \delta_\alpha j^{1/p}]}]^{p/2} < \infty$

(c) $\sum_\alpha \sum_j \int_{\delta_\alpha}^{\infty} t^{p-1} P\{|X_\alpha| > t j^{1/p}\}\, dt < \infty$,

where $\delta_\alpha = \inf[\delta: \sum_j P\{|X_\alpha| > \delta j^{1/p}\} \leqq 1/8 \cdot 3^2]$. Since $\sum_j P\{|X_\alpha|^p/\delta^p > j\} \approx E|X_\alpha|^p/\delta^p$, it follows that $\delta_\alpha \approx \|X_\alpha\|_p$. We then conclude that the condition $\sum_\alpha \|X_\alpha\|_p^p < \infty$ is equivalent to (a) and implies (b) (for the last statement interchange $\sum_j$ and E). As for (c) we have, since $|X_\alpha| \leqq 1$:

$$\begin{aligned}
&\sum_\alpha \sum_{j=1}^{\infty} \int_{\delta_\alpha}^{\infty} t^{p-1} P\{|X_\alpha| > t j^{1/p}\}\, dt \\
&\quad = \sum_\alpha \int_0^1 \sum_j I_{[1 \leqq j \leqq u^p/\delta_\alpha^p]} j^{-1} u^{p-1} P\{|X_\alpha| > u\}\, du \\
&\quad \approx \sum_\alpha \int_0^1 \left(1 + \ln_+ \frac{u^p}{\delta_\alpha^p}\right) u^{p-1} P\{|X_\alpha| > u\}\, du,
\end{aligned}$$

the convergence of which is equivalent to (6.24).

It is easy (but tedious) to check that both the lemma and the proposition are also true for l_p, $0 < p < 1$.