A Schauder basis for $L_1(0,\infty)$ consisting of non-negative functions*

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Abstract

We construct a Schauder basis for L_1 consisting of non-negative functions and investigate unconditionally basic and quasibasic sequences of non-negative functions in L_p.

1 Introduction

In [5], Powell and Spaeth investigate non-negative sequences of functions in L_p, $1 \le p < \infty$, that satisfy some kind of basis condition, with a view to determining whether such a sequence can span all of L_p. They prove, for example, that there is no unconditional basis or even unconditional quasibasis (frame) for L_p consisting of non-negative functions. On the other hand, they prove that there are non-negative quasibases and non-negative M-bases for L_p. The most important question left open by their investigation is whether there is a (Schauder) basis for L_p consisting of non-negative functions. In section 2 we show that there is basis for L_1 consisting of non-negative functions.

In section 3 we discuss the structure of unconditionally basic non-negative normalized sequences in L_p. The main result is that such a sequence is equivalent to the unit vector basis of ℓ_p. We also prove that the closed span

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in L_p of any unconditional quasibasic sequence embeds isomorphically into ℓ_p.

We use standard Banach space theory, as can be found in [4] or [1]. Let us just mention that L_p is $L_p(0,\infty)$, but inasmuch as this space is isometrically isomorphic under an order preserving operator to $L_p(\mu)$ for any separable purely non-atomic measure μ, our choice of $L(0,\infty)$ rather than e.g. $L_p(0,1)$ is a matter of convenience. Again as a matter of convenience, in the last part of Section 3 we revert to using $L_p(0,1)$ as a model for L_p.

2 A Schauder basis for $L_1(0,\infty)$ consisting of non-negative functions

For $j=1,2,\dots$ let $\{h^j_{n,i}\}_{n=0,i=1}^{\infty\ \ 2^n}$ be the mean zero L_1 normalized Haar functions on the interval $(j-1,j)$. That is, for $n=0,1,\dots,$ $i=1,2,\dots,2^n$,

$$h^j_{n,i}(t)=\begin{cases} 2^n & j-1+\frac{2i-2}{2^{n+1}}<t<j-1+\frac{2i-1}{2^{n+1}}\\ -2^n & j-1+\frac{2i-1}{2^{n+1}}<t<j-1+\frac{2i}{2^{n+1}}\\ 0 & otherwise\end{cases}$$

The system $\{h^j_{n,i}\}_{n=0,i=1,j=1}^{\infty\ \ 2^n\ \ \infty}$, in any order which preserves the lexicographic order of $\{h^j_{n,i}\}_{n=0,i=1}^{\infty\ \ 2^n}$ for each j, constitutes a basis for the subspace of $L_1(0,\infty)$ consisting of all functions whose restriction to each interval $(j-1,j)$ have mean zero. To simplify notation, for each j we shall denote by $\{h^j_i\}_{i=1}^\infty$ the system $\{h^j_{n,i}\}_{n=0,i=1}^{\infty\ \ 2^n}$ in its lexicographic order. We shall also denote by $\{h_i\}_{i=1}^\infty$ the union of the systems $\{h^j_i\}_{i=1}^\infty$, $j=1,2,\dots,$ in any order that respects the individual orders of each of the $\{h^j_i\}_{i=1}^\infty$.

Let π be any permutation of the natural numbers and for each $i\in\mathbb{N}$ let F_i be the two dimensional space spanned by $2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|$ and h_i.

Proposition 1 *$\sum_{i=1}^\infty F_i$ is an FDD of $\overline{\mathrm{span}}^{L_1}\{F_i\}_{i=1}^\infty$.*

Proof: The assertion will follow from the following inequality, which holds for all scalars $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$,

$$\begin{aligned}\tfrac{1}{2}\sum_{i=1}^\infty|a_i|+\tfrac{1}{8}\|\sum_{i=1}^\infty b_ih_i\| &\le \|\sum_{i=1}^\infty a_i(2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|)+\sum_{i=1}^\infty b_ih_i\|\\ &\le 3\sum_{i=1}^\infty|a_i|+\|\sum_{i=1}^\infty b_ih_i\|.\end{aligned} \tag{1}$$

The right inequality in (1) follows easily from the triangle inequality. As for the left inequality, notice that the conditional expectation projection onto the closed span of $\{\mathbf{1}_{(i-1,i)}\}_{i=1}^{\infty}$ is of norm one and the complementary projection, onto the closed span of $\{h_i\}_{i=1}^{\infty}$, is of norm 2. It follows that

$$\|\sum_{i=1}^{\infty} a_i(2\mathbf{1}_{(\pi(i)-1,\pi(i))}) + \sum_{i=1}^{\infty} b_i h_i\| \geq \max\{2\sum_{i=1}^{\infty}|a_i|, \frac{1}{2}\|\sum_{i=1}^{\infty} b_i h_i\|\}.$$

Since $\|\sum_{i=1}^{\infty} a_i|h_i|\| \leq \sum_{i=1}^{\infty}|a_i|$, we get

$$\|\sum_{i=1}^{\infty} a_i(2\mathbf{1}_{(\pi(i)-1,\pi(i))} + |h_i|) + \sum_{i=1}^{\infty} b_i h_i\| \geq \max\{\sum_{i=1}^{\infty}|a_i|, \frac{1}{4}\|\sum_{i=1}^{\infty} b_i h_i\|\}$$

from which the left hand side inequality in (1) follows easily. ■

Proposition 2 *Let π be any permutation of the natural numbers and for each $i \in \mathbb{N}$ let F_i be the two dimensional space spanned by $2\mathbf{1}_{(\pi(i)-1,\pi(i))} + |h_i|$ and h_i. Then $\overline{\text{span}}^{L_1}\{F_i\}_{i=1}^{\infty}$ admits a basis consisting of non-negative functions.*

Proof: In view of Proposition 1 it is enough to show that each F_i has a two term basis consisting of non-negative functions and with uniform basis constant. Put $x_i = 2\mathbf{1}_{(\pi(i)-1,\pi(i))} + |h_i| + h_i$ and $y_i = 2\mathbf{1}_{(\pi(i)-1,\pi(i))} + |h_i| - h_i$. Then clearly $x_i, y_i \geq 0$ everywhere and $\|x_i\| = \|y_i\| = 3$. We now distinguish two cases: If $\mathbf{1}_{(\pi(i)-1,\pi(i))}$ is disjoint from the support of h_i then, for all scalars a, b,

$$\|ax_i + by_i\| \geq \|a(|h_i| + h_i) + b(|h_i| - h_i)\| = |a| + |b|.$$

If the support of h_i is included in $(\pi(i) - 1, \pi(i))$, Let 2^{-s} be the size of that support, $s \geq 0$. Then for all scalars a, b,

$$\begin{aligned} \|ax_i + by_i\| &\geq \|a(|h_i| + h_i) + b(|h_i| - h_i) + 2(a+b)\mathbf{1}_{\text{supp}(h_i)}\| \\ &= 2^{-s-1}(|(2^{s+1}+2)a + 2b| + |(2^{s+1}+2)b + 2a| \geq \max\{|a|, |b|\}. \end{aligned}$$

■

Theorem 1 *$L_1(0,\infty)$, and consequently any separable L_1 space, admits a Schauder basis consisting of non-negative functions.*

Proof: When choosing the order on $\{h_i\}$ we can and shall assume that $h_1 = h^1_{0,1}$; i.e., the first mean zero Haar function on the interval $(0, 1)$. Let π be any permutation of $\mathbb{N}$ such that $\pi(1) = 1$ and for $i > 1$, if $h_i = h^j_{n,k}$ for some n, k, and j then $\pi(i) > j$. It follows that except for $i = 1$ the support of h_i is disjoint from the interval $(\pi(i) - 1, \pi(i))$. It is easy to see that such a permutation exists. We shall show that under these assumptions $\sum_{i=1}^{\infty} F_i$ spans $L_1(0,\infty)$ and, in view of Proposition 2, this will prove the theorem for $L_1(0,\infty)$. First, since $\pi(1) = 1$ we get that $3\mathbf{1}_{(0,1)} = 2\mathbf{1}_{(\pi(1)-1,\pi(1))} + |h_1| \in F_1$, and since all the mean zero Haar functions on $(0, 1)$ are clearly in $\sum_{i=1}^{\infty} F_i$, we get that $L_1(0, 1) \subset \sum_{i=1}^{\infty} F_i$.

Assume by induction that $L_1(0, j) \subset \sum_{i=1}^{\infty} F_i$. Let l be such that $\pi(l) = j+1$. By our assumption on π, the support of h_l is included in $(0, j)$, and so by the induction hypothesis, $|h_l| \in \sum_{i=1}^{\infty} F_i$. Since also $2\mathbf{1}_{(j,j+1)} + |h_l| \in \sum_{i=1}^{\infty} F_i$ we get that $\mathbf{1}_{(j,j+1)} \in \sum_{i=1}^{\infty} F_i$. Since the mean zero Haar functions on $(j, j+1)$ are also in $\sum_{i=1}^{\infty} F_i$ we conclude that $L_1(0, j + 1) \subset \sum_{i=1}^{\infty} F_i$.

This finishes the proof for $L_1(0,\infty)$. Since every separable L_1 space is order isometric to one of the spaces ℓ_1^k, $k = 1, 2, \ldots, \ell_1, L_1(0,\infty), L_1(0,\infty) \bigoplus_1 \ell_1^k$, $k = 1, 2, \ldots,$ or $L_1(0,\infty) \bigoplus_1 \ell_1$, and since the discrete L_1 spaces ℓ_1^k, $k = 1, 2, \ldots,$ and ℓ_1 clearly have non-negative bases, we get the conclusion for any separable L_1 space. ∎

3 Unconditional non-negative sequences in L_p

Here we prove

Theorem 2 *Suppose that $\{x_n\}_{n=1}^{\infty}$ is a normalized unconditionally basic sequence of non-negative functions in L_p, $1 \le p < \infty$. Then $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p.*

Proof: First we give a sketch of the proof, which should be enough for experts in Banach space theory. By unconditionality, we have for all coefficients a_n that $\|\sum_n a_n x_n\|_p$ is equivalent to the square function $\|(\sum_n |a_n|^2 x_n^2)^{1/2}\|_p$, and, by non-negativity of x_n, is also equivalent to $\|\sum_n |a_n| x_n\|_p$. Thus by trivial interpolation when $1 \le p \le 2$, and by extrapolation when $2 < p < \infty$, we see that $\|\sum_n a_n x_n\|_p$ is equivalent to $\|(\sum_n |a_n|^p x_n^p)^{1/p}\|_p = (\sum_n |a_n|^p)^{1/p}$.

We now give a formal argument for the benefit of readers who are not familiar with the background we assumed when giving the sketch. Let K be the unconditional constant of $\{x_n\}_{n=1}^\infty$. Then

$$\begin{aligned} K^{-1}\|\sum_{n=1}^N a_n x_n\|_p &\le B_p\|(\sum_{n=1}^N |a_n|^2 x_n^2)^{1/2}\|_p \\ &\le B_p\|\sum_{n=1}^N |a_n| x_n\|_p \le B_p K\|\sum_{n=1}^N a_n x_n\|_p, \end{aligned} \tag{2}$$

where the first inequality is obtained by integrating against the Rademacher functions (see, e.g., [4, Theorem 2.b.3]). The constant B_p is Khintchine's constant, so $B_p = 1$ for $p \le 2$ and B_p is of order $\sqrt{p}$ for $p > 2$. If $1 \le p \le 2$ we get from (2)

$$K^{-1}\|\sum_{n=1}^N a_n x_n\|_p \le \|(\sum_{n=1}^N |a_n|^p x_n^p)^{1/p}\|_p \le K\|\sum_{n=1}^N a_n x_n\|_p. \tag{3}$$

Since $\|(\sum_{n=1}^N |a_n|^p x_n^p)^{1/p}\|_p = (\sum_{n=1}^N |a_n|^p)^{1/p}$, this completes the proof when $1 \le p \le 2$. When $2 < p < \infty$, we need to extrapolate rather than do (trivial) interpolation. Write $1/2 = \theta/1 + (1-\theta)/p$. Then

$$\begin{aligned} (KB_p)^{-1}\|\sum_{n=1}^N a_n x_n\|_p &\le \|(\sum_{n=1}^N |a_n|^2 x_n^2)^{1/2}\|_p \\ &\le \|\sum_{n=1}^N |a_n| x_n\|_p^\theta \|(\sum_{n=1}^N |a_n|^p x_n^p)^{1/p}\|_p^{1-\theta} \\ &\le K\|\sum_{n=1}^N a_n x_n\|_p^\theta (\sum_{n=1}^N |a_n|^p)^{(1-\theta)/p}, \quad \text{so that} \end{aligned} \tag{4}$$

$$(K^2 B_p)^{(-1)/(1-\theta)}\|\sum_{n=1}^N a_n x_n\|_p \le (\sum_{n=1}^N |a_n|^p)^{1/p} \le K\|\sum_{n=1}^N a_n x_n\|_p. \quad \blacksquare$$

As stated, Theorem 2 gives no information when $p = 2$ because every normalized unconditionally basic sequence in a Hilbert space is equivalent to the unit vector basis of ℓ_2. However, if we extrapolate slightly differently in the above argument (writing $1/2 = \theta/1 + (1-\theta)/\infty$) we see that, no matter what p is, $\|\sum_{n=1}^N a_n x_n\|_p$ is also equivalent to $\|\max_n |a_n| x_n\|_p$. From this one can deduce e.g. that only finitely many Rademachers can be in the closed span of $\{x_n\}_{n=1}^\infty$; in particular, $\{x_n\}_{n=1}^\infty$ cannot be a basis for L_p even when $p = 2$. However, the proof given in [5] that a normalized unconditionally basic sequence of non-negative functions $\{x_n\}_{n=1}^\infty$ in L_p cannot span L_p actually shows that only finitely many Rademachers can be in the closed span of $\{x_n\}_{n=1}^\infty$. This is improved in our last result, which shows that the closed span of an unconditionally non-negative quasibasic sequence in $L_p(0,1)$ cannot contain any strongly embedded infinite dimensional subspace (a subspace X of $L_p(0,1)$ is said to be strongly embedded if the $L_p(0,1)$ norm is equivalent to the $L_r(0,1)$ norm on X for some–or, equivalently, for all– $r < p$; see e.g. [1, p. 151]). The main work for proving this is contained in Lemma 1.

Before stating Lemma 1, we recall that a quasibasis for a Banach space X is a sequence $\{f_n, g_n\}_{n=1}^\infty$ in $X \times X^*$ such that for each x in X the series $\sum_n \langle g_n, x\rangle f_n$ converges to x. (In [5] a sequence $\{f_n\}_{n=1}^\infty$ in X is a called a quasibasis for X provided there exists such a sequence $\{g_n\}_{n=1}^\infty$. Since the sequence $\{g_n\}_{n=1}^\infty$ is typically not unique, we prefer to specify it up front.) The quasibasis $\{f_n, g_n\}_{n=1}^\infty$ is said to be unconditional provided that for each x in X the series $\sum_n \langle g_n, x\rangle f_n$ converges unconditionally to x. One then gets from the uniform boundedness principle (see, e.g., [5, Lemma 3.2]) that there is a constant K so that for all x and all scalars a_n with $|a_n| \le 1$, we have $\| \sum_n a_n \langle g_n, x\rangle f_n\| \le K\|x\|$. A sequence $\{f_n, g_n\}_{n=1}^\infty$ in $X \times X^*$ is said to be [unconditionally] quasibasic provided $\{f_n, h_n\}_{n=1}^\infty$ is an [unconditional] quasibasis for the closed span $[f_n]$ of $\{f_n\}_{n=1}^\infty$, where h_n is the restriction of g_n to $[f_n]$.

Lemma 1 *Suppose that $\{f_n, g_n\}_{n=1}^\infty$ is an uncondtionally quasibasic sequence in $L_p(0,1)$, $1 < p < \infty$ with each f_n non-negative. If $\{y_n\}_{n=1}^\infty$ is a normalized weakly null sequence in $[f_n]$, then $\|y_n\|_1 \to 0$ as $n \to \infty$.*

Proof: If the conclusion is false, we get a normalized weakly null sequence $\{y_n\}_{n=1}^\infty$ in $[f_n]$ and a $c > 0$ so that for all n we have $\|y_n\|_1 > c$.

By passing to a subsequence of $\{y_n\}_{n=1}^\infty$, we can assume that there are integers $0 = m_1 < m_2 < \dots$ so that for each n,

$$\sum_{k=1}^{m_n} |\langle g_k, y_n\rangle| < 2^{-n-3}c \quad \text{and} \quad \| \sum_{k=m_{n+1}+1}^{\infty} |\langle g_k, y_n\rangle| f_n\|_p < 2^{-n-3}c. \qquad (5)$$

Effecting the first inequality in (5) is no problem because $y_n \to 0$ weakly, but the second inequality perhaps requires a comment. Once we have a

y_n that satisfies the first inquality in (5), from the unconditional convergence of the expansion of y_n and the non-negativity of all f_k we get that $\| \sum_{k=N}^{\infty} |\langle g_k, y_n\rangle| f_k\|_p \to 0$ as $n \to \infty$, which allows us to select m_{n+1} to satisfy the second inequality in (5).

Since $\|f_n\|_1 > c$, from (5) we also have for every n that

$$\| \sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n\rangle| f_n\|_1 \geq \| \sum_{k=m_n+1}^{m_{n+1}} \langle g_k, y_n\rangle f_n\|_1 \geq c/2. \tag{6}$$

Since L_p has an unconditional basis, by passing to a further subsequence we can assume that $\{y_n\}_{n=1}^{\infty}$ is unconditionally basic with constant K_p. Also, L_p has type s, where $s = p \wedge 2$ (see [1, Theorem 6.2.14]), so for some constant K'_p we have for every N the inequality

$$\| \sum_{n=1}^{N} y_n\|_p \leq K'_p N^{1/s}. \tag{7}$$

On the other hand, letting $\delta_k = \operatorname{sign}\langle g_k, y_n\rangle$ when $m_n + 1 \leq k \leq m_{n+1}$, $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
K_p\| \sum_{n=1}^{N} y_n\|_p &\geq K_p\| \sum_{n=1}^{N}\sum_{k=1}^{\infty} \delta_k\langle g_k, y_n\rangle f_k\|_p \\
&\geq \| \sum_{n=1}^{N}\sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n\rangle| f_k\|_p - \| \sum_{n=1}^{N}\sum_{k\notin[m_n+1, m_{n+1}]} \delta_k\langle g_k, y_n\rangle f_k\|_p \\
&\geq \| \sum_{n=1}^{N}\sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n\rangle| f_k\|_1 - \| \sum_{n=1}^{N}\sum_{k\notin[m_n+1, m_{n+1}]} |\langle g_k, y_n\rangle| f_k\|_p \\
&\geq \sum_{n=1}^{N} \| \sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n\rangle| f_k\|_1 \\
&\quad - \sum_{n=1}^{N}\left(\sum_{k=1}^{m_n} |\langle g_k, y_n\rangle| + \| \sum_{k=m_{n+1}+1}^{\infty} |\langle g_k, y_n\rangle| f_n\|_p \right) \\
&\geq Nc/2 - c/4 \qquad \text{by (6) and (5)}
\end{aligned} \tag{8}$$

This contradicts (7). ■

Theorem 3 *Suppose that $\{f_n, g_n\}_{n=1}^{\infty}$ is an unconditional quasibasic sequence in $L_p(0,1)$, $1 \leq p < \infty$, and each f_n is non-negative. Then the closed span $[f_n]$ of $\{f_n\}_{n=1}^{\infty}$ embeds isomorphically into ℓ_p.*

Proof: The case $p = 1$ is especially easy: There is a constant K so that for each y in $[f_n]$

$$\|y\|_1 \leq \| \sum_{n=1}^{\infty} |\langle g_k, y\rangle| f_n\|_1 \leq K\|y\|_1, \tag{9}$$

hence the mapping $y \mapsto \{\langle g_k, y\rangle\}_{k=1}^{\infty}$ is an isomorphism from $[f_n]$ into ℓ_1.

So in the sequel assume that $p > 1$. From Lemma 1 and standard arguments (see, e.g., [1, Theorem 6.4.7]) we have that every normalized weakly null sequence in $[f_n]$ has a subsequence that is an arbitrarliy small perturbation of a disjoint sequence and hence the subsequence is $1+\epsilon$-equivalent to the unit vector basis for ℓ_p. This implies that $[f_n]$ embeds isomorphically into ℓ_p (see [3] for the case $p > 2$ and [2, Theorems III.9, III.1, and III.2] for the case $p < 2$). ■

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