Lipschitz p-summing Operators*

Jeffrey D. Farmer and William B. Johnson†

Abstract

The notion of Lipschitz p-summing operator is introduced. A non linear Pietsch factorization theorem is proved for such operators and it is shown that a Lipschitz p-summing operator that is linear is a p-summing operator in the usual sense.

1 Introduction

In this note we introduce a natural non linear version of p-summing operator, which we call Lipschitz p-summing operator. In section 2 we prove a non linear version of the Pietsch factorization theorem, show by example that the strong form of the Pietsch domination theorem is not true for Lipschitz p-summing operators, and make a few other remarks about these operators. In section 3 we "justify" our nomenclature by proving that for a linear operator, the Lipschitz p-summing norm is the same as the usual p-summing norm. Finally, in section 4 we raise some problems which we think are interesting.

2 Pietsch factorization

The *Lipschitz p-summing* ($1 \leq p < \infty$) *norm*, $\pi_p^L(T)$, of a (possibly non linear) mapping $T : X \to Y$ between metric spaces is the smallest constant C so that for all (x_i), (y_i) in X and all positive reals a_i

$$\sum a_i \|Tx_i - Ty_i\|^p \leq C^p \sup_{f \in B_{X^\#}} \sum a_i |f(x_i) - f(y_i)|^p \tag{1}$$

Here $B_{X^\#}$ is the unit ball of $X^\#$, the Lipschitz dual of X, i.e., $X^\#$ is the space of all real valued Lipschitz functions under the (semi)-norm $\mathrm{Lip}(\cdot)$; and $\|x - y\|$ is the distance from x to y in Y. We follow the usual convention of considering X as a pointed metric space by designating a special point $0 \in X$ and identifying $X^\#$ with the Lipschitz

*AMS subject classification: 46B28,46T99, 47H99,47L20. Key words: p-summing operator, absolutely summing operator.

†Supported in part by NSF DMS-0503688

functions on X that are zero at 0. With this convention $(X^{\#}, \mathrm{Lip}(\cdot))$ is a Banach space and $B_{X^{\#}}$ is a compact Hausdorff space in the topology of pointwise convergence on X.

Notice that the definition is the same if we restrict to $a_i = 1$. Indeed, by approximation it is enough to consider rational a_i and thus, by clearing denominators, integer a_i. Then, given a_i, x_i, and y_i, consider the new collection of vectors in which the pair (x_i, y_i) is repeated a_i times. (This observation was made with M. Mendel and G. Schechtman.)

It is clear that π_p^L has the ideal property; i.e., $\pi_p^L(ATB) \le \mathrm{Lip}(A)\pi_p^L(T)\mathrm{Lip}(B)$ whenever the compositions make sense. Also, if Y is a Banach space, the space of Lipschitz p-summing maps from any metric space into Y is a Banach space under the norm π_p^L.

If T is a linear operator, it is clear that $\pi_p^L(T) \le \pi_p(T)$, where $\pi_p(\cdot)$ is the usual p-summing norm [5, p. 31]. In section 3 we prove that the reverse inequality is true.

We begin with a Pietsch factorization theorem for Lipschitz p-summing operators.

Theorem 1 *The following are equivalent for a mapping $T : X \to Y$ between metric spaces and $C \ge 0$.*

1. $\pi_p^L(T) \le C$.

2. There is a probability μ on $B_{X^{\#}}$ such that

$$\|Tx - Ty\|^p \le C^p \int_{B_{X^{\#}}} |f(x) - f(y)|^p \, d\mu(f).$$

(Pietsch domination.)

3. For some (or any) isometric embedding J of Y into a 1-injective space Z, there is a factorization

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\ X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & Z \end{tikzcd}$$

with μ a probability and $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \le C$.

(Pietsch factorization.)

Proof: That (2) implies (3) is basically obvious: Let $A : X \to L_\infty(\mu)$ be the natural isometric embedding composed with the formal identity from $C(B_{X^{\#}})$ into $L_\infty(\mu)$. Then (2) says that the Lipschitz norm of B restricted to $I_{\infty,p}AX$ is bounded by C, which is just (3). (We have used implicitly the well known fact that every metric space embeds into $\ell_\infty(\Gamma)$ for some set Γ and that, by the non linear Hahn-Banach theorem, $\ell_\infty(\Gamma)$ is 1-injective. See Lemma 1.1 in [3].)

For (3) implies (1), use
$\pi_p^L(T) = \pi_p^L(JT) \le \mathrm{Lip}(A)\pi_p^L(I_{\infty,p})\mathrm{Lip}(B) \le \mathrm{Lip}(A)\pi_p(I_{\infty,p})\mathrm{Lip}(B) = \mathrm{Lip}(A)\mathrm{Lip}(B)$.

The proof of the main implication, that (1) implies (2), is like the proof of the (linear) Pietsch factorization theorem (see, e.g., [5, p. 44]). Suppose $\pi_p^L(T) = 1$. Let Q be the convex cone in $C(B_{X^\#})$ consisting of all positive linear combinations of functions of the form $\|Tx - Ty\| - C^p|f(x) - f(y)|^p$, as x and y range over X. Condition (1) says that Q is disjoint from the the positive cone $P = \{F \in C(B_{X^\#}) \mid F(f) > 0 \ \forall f \in X^\#\}$, which is an open convex subset of $C(B_{X^\#})$. Thus by the separation theorem and the Riesz representation theorem there is a finite signed Baire measure μ on $B_{X^\#}$ and a real number c so that for all $G \in Q$ and $F \in P$, $\int_{X^\#} G\, d\mu \le c < \int_{X^\#} F\, d\mu$. Since $0 \in Q$ and all positive constants are in P, we see that $c = 0$, and since $\int_{X^\#} \cdot \, d\mu$ is positive on the positive cone P of $C(B_{X^\#})$, the signed measure μ is a positive measure, which we can assume by rescaling is a probability measure. It is clear that the inequality in (2) is satisfied. ∎

It is worth noting that the conditions in Theorem 1 are also equivalent to

4. *There is a probability μ on K, the closure in the topology of pointwise convergence on X of the extreme points of $B_{X^\#}$, so that*

$$\|Tx - Ty\|^p \le C^p \int_K |f(x) - f(y)|^p \, d\mu(f).$$

The proof that (1) implies (4) is the same as the proof that (1) implies (2) since the supremum on the right side of (1), the definition of the Lipschitz p-summing norm, is the same as

$$\sup_{f \in K} \sum a_i |f(x_i) - f(y_i)|^p.$$

One immediate consequence of Theorem 1 is that $\pi_p^L(T)$ is a monotonely decreasing function of p. Another consequence is that there is a version of Grothendieck's theorem (that every linear operator from an L_1 space to a Hilbert space is 1-absolutely summing). In the category of metric spaces with Lipschitz mappings as morphisms, weighted trees play a role analogous to that of L_1 in the linear theory. In particular, every finite weighted tree has the lifting property, which is to say that if X is a finite weighted tree, $T : X \to Y$ is a Lipschitz mapping from X into a metric space Y, and $Q : Z \to Y$ is a 1-Lipschitz quotient mapping in the sense of [2], [7], then for each $\varepsilon > 0$ there is a mapping $S : X \to Z$ so that $\mathrm{Lip}(S) \le \mathrm{Lip}(T) + \varepsilon$ and $T = QS$. Letting Y be a Hilbert space and Z an L_1 space, we see from Grothendieck's theorem and the ideal property of π_1^L that if every finite subset of X is contained in a finite subset of X that is a weighted tree (in particular, if X is a tree or a metric tree–see [7]), then $\pi_1^L(T) \le K_G \mathrm{Lip}(T)$, where K_G is Grothendieck's constant. Here we use the obvious fact that $\pi_p^L(T : X \to Y)$ is the supremum of $\pi_p^L(T_{|K})$ as K ranges over finite subsets of X.

The strong form of the Pietsch domination theorem says that if X is a subspace of $C(K)$ for some compact Hausdorff space K, and T is a p-summing linear operator with domain X, then there is a probability measure μ on K so that for all $x \in X$, $\|Tx\|^p \le \pi_p(T)^p \int_K |x(t)|^p \, d\mu(t)$. It is easy to see that there is not a non linear version of this result. Let D_n be the discrete metric space with n points so that the distance between any two distinct points is one. We can embed D_n into $C(\{-1, 1\}^n)$ in two essentially different ways. First, if $D_n = \{x_1, \dots, x_n\}$, let $f(x_k) = \frac{1}{2} r_k$, where r_k is the projection onto the kth coordinate. The image of this set under the canonical injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \mu)$ with μ the uniform probability on $\{-1,1\}^n$ is a discrete set with the p-th power of all distances one-half. This shows that the identity on D_n has Lipschitz p-summing norm at most two. Secondly, let $g(k)$, $1 \le k \le n$, be disjointly supported unit vectors in $C(\{-1,1\}^n)$. Then for any probability measure ν on $\{-1,1\}^n$, the injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \nu)$ shrinks the distance between some pair of the $g(k)$'s to at most $(2/n)^{1/p}$.

Incidentally, $\pi_p^L(I_{D_n})$ tends to $2^{\frac{1}{p}}$ as $n \to \infty$ and can be computed exactly. To see this, note that the extreme points, K_n, of $B_{D_n^\#}$ are of the form $\pm\chi_A$ with A a non empty subset of $D_n \sim \{0\}$. This can be calculated directly or deduced from Theorem 1 in [6]. We calculate $\pi_p^L(I_{D_n})$ in the (easier) case that n is even. Define a probability μ on K_n by letting μ be the uniform measure on $J_{n/2} := \{\chi_A : |A| = n/2, A \subset D_n \sim \{0\}\}$ (so that $\mu(e) = 0$ for elements e of $K_n \sim J_{n/2}$). Then for each pair of distinct points x and y in D_n, $\int_{K_n} |f(x) - f(y)|^p \, d\mu(f) = \frac{n}{2(n-1)}$, so that $\pi_p^L(I_{D_n}) \le (2 - \frac{2}{n})^{\frac{1}{p}}$. To see that μ is a Pietsch measure for I_{D_n}, let ν be any Pietsch probability for I_{D_n} on K_n. We can clearly assume that ν is supported on the positive elements in K_n. By averaging ν against the permutations of D_n which fix 0, which is a group of isometries on D_n, we get another Pietsch probability for I_{D_n} (which we continue to denote by ν) so that if we condition ν on $J_k := \{\chi_A : |A| = k, A \subset D_n \sim \{0\}\}$, $1 \le k \le n-1$, the resulting probability ν_k on J_k is the uniform probability. A trivial calculation shows that for x, y in $D_n \sim \{0\}$, $\int_{J_k} |f(x) - f(y)|^p \, d\nu_k(f) \le \frac{n}{2(n-1)}$. This proves that μ is a Pietsch measure for I_{D_n} and hence $\pi_p^L(I_{D_n}) = (2 - \frac{2}{n})^{\frac{1}{p}}$.

Our final comment on Lipschitz 1-summing operators is that the concept has appeared in the literature even if the definition is new. In [4], Bourgain proved that every n point metric space can be embedded into a Hilbert space with distortion at most $C \log n$, where C is an absolute constant. In fact, he really proved the much stronger result that $\pi_1^L(I_X) \le C \log n$ if I_X is the identity mapping on an n point space X by making use of a special embedding of X into a space $C(K_X)$ with K_X a finite metric space and constructing a probability on K_X. Moreover, Bourgain's construction has occasionally been used in the computer science literature. The strong form of Bourgain's theorem is also used in [8] to prove an inequality that is valid for all metric spaces.

3 Linear operators

In this section we show that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm. This justifies that the notion of Lipschitz p-summing operator is really a generalization of the concept of linear p-summing operator.

Theorem 2 *Let u be a bounded linear operator from X into Y and $1 \le p < \infty$. Then $\pi_p^L(u) = \pi_p(u)$.*

Proof: Note that we can assume, without loss of generality, that $\dim Y \le \dim X = N < \infty$. Indeed, it is clear from the definition that $\pi_p^L(u)$ is the supremum of $\pi_p^L(u_{|E})$ as E ranges over finite dimensional subspaces of X and similarly for $\pi_p^L(u)$. That we can assume $\dim Y \le \dim X$ is clear from the linearity of u.

Since $\dim Y \le N$, there is an embedding J of Y into ℓ_∞^m with $m \le (\frac{3}{\varepsilon})^N$ so that $\|J\| = 1$ and $\|J^{-1}\| \le 1 + \varepsilon$. We then get the following non linear Pietsch factorization:

$$\begin{array}{ccccc}
L_\infty(\mu) & & \xrightarrow{i_{\infty,p}} & & L_p(\mu) \\
\alpha \uparrow & & & & \downarrow \beta \\
X & \xrightarrow{u} & Y & \xrightarrow{J} & \ell_\infty^m
\end{array}$$

where $\mathrm{Lip}(\alpha) = 1$, $\mathrm{Lip}(\beta) \le \pi_p^L(Ju) \le \pi_p^L(u)$. We can also assume, without loss of generality, that the probability μ is a separable measure.

We now use some non linear theory that can be found in the book [3].

1. The mapping α is weak* differentiable almost everywhere. This means that for (Lebesgue) almost every x_0 in X, there is a linear operator $D_{x_0}^{w^*}(\alpha)\colon X \to L_\infty(\mu)$ so that for all $f \in L_1(\mu)$ and for every $y \in X$,

$$\lim_{t \to 0} \left\langle \frac{\alpha(x_0 + ty) - \alpha(x_0)}{t}, f \right\rangle = \langle D_{x_0}^{w^*}(\alpha)(y), f \rangle.$$

2. The operator $i_{\infty,p}\alpha$ is differentiable almost everywhere. This means that for almost every x_0 in X, there is a linear operator $D_{x_0}(i_{\infty,p}\alpha)\colon X \to L_p(\mu)$ so that

$$\sup_{\|y\| \le 1} \left\| \frac{i_{\infty,p}\alpha(x_0 + ty) - i_{\infty,p}\alpha(x_0)}{t} - D_{x_0}(i_{\infty,p}\alpha)(y) \right\|_p \to 0 \quad \text{as} \quad t \to 0.$$

When $1 < p < \infty$, statement (2) follows from the reflexivity of L_p (see [3, Corollary 5.12 & Proposition 6.1]). For $p = 1$, just use (2) for $p = 2$ and compose with $i_{2,1}$.

The mapping $i_{\infty,p}$ is weak* to weak continuous, so $D_{x_0}(i_{\infty,p}\alpha) = i_{\infty,p} D_{x_0}^{w^*}(\alpha)$ whenever both derivatives exist. Since they both exist almost everywhere, by making several translations we can assume without loss of generality that this equation is true for $x_0 = 0$ and also that $\alpha(0) = 0$.

Next we show that in the factorization diagram the non linear map α can be replace by the linear operator $D_0^{w^*}(\alpha)$ by constructing a mapping $\tilde{\beta} : L_p(\mu) \to \ell_\infty^m$ so that $\tilde{\beta} i_{\infty,p} D_0^{w^*}(\alpha) = Ju$ and $\mathrm{Lip}(\tilde{\beta}) \le \mathrm{Lip}(\beta)$. To do this, define $\beta_n : L_p(\mu) \to \ell_\infty^m$ by $\beta_n(y) := n\beta(\frac{y}{n})$ and note that $\mathrm{Lip}(\beta_n) = \mathrm{Lip}(\beta)$. We have for each x in X

$$\begin{aligned}
\|Ju(x) - \beta_n i_{\infty,p} D_0^{w^*}(\alpha)(x)\| &= \|\beta_n n i_{\infty,p}\alpha(x/n) - \beta_n D_0(i_{\infty,p}\alpha)(x)\| \\
&\le \mathrm{Lip}(\beta) \|n i_{\infty,p}\alpha(x/n) - D_0(i_{\infty,p}\alpha)(x)\|
\end{aligned}$$

which tends to zero as $n \to \infty$. For $\tilde{\beta}$ we can take any cluster point of β_n in the space of functions from $L_p(\mu)$ into ℓ_∞^m; such exist because β_n is uniformly Lipschitz and $\beta_n(0) = 0$.

Summarizing, we see that we have a factorization

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{i_{\infty,p}} & & L_p(\mu) \\ \tilde{\alpha}\uparrow & & & & \downarrow\tilde{\beta} \\ X & \xrightarrow{u} & Y & \xrightarrow{J} & \ell_\infty^m \end{array}$$

with $\tilde{\alpha}$ linear, $\|\tilde{\alpha}\| \le \mathrm{Lip}(\alpha)$, and $\mathrm{Lip}(\tilde{\beta}) \le \mathrm{Lip}(\beta)$.

The final step involves replacing $\tilde{\beta}$ with a linear operator. Since the restriction of $\tilde{\beta}$ to the linear subspace $i_{\infty,p}\tilde{\alpha}[X]$ is linear and ℓ_∞^m is reflexive, this follows from [3, Theorem 7.2], which is proved by a simple invariant means argument.

4 Open problems and concluding remarks

Problem 1 *Is there a composition formula for Lipschitz p-summing operators? That is, do we have $\pi_p^L(TS) \le \pi_r^L(T)\pi_s^L(S)$, when $\frac{1}{p} \le (\frac{1}{r} + \frac{1}{s}) \wedge 1$?*

Say that a Lipschitz mapping $T : X \to Y$ is Lipschitz p-integral if it satisfies a factorization diagram as in condition (3) of Theorem 1, except with J being the canonical isometry from Y into $(Y^{\#})^*$. We then define the Lipschitz p-integral norm $I_p^L(T)$ of T to be the infimum of $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B)$, the infimum being taken over all such factorizations. When T is a linear operator, this is the same as the usual p-integral norm of T. Indeed, in this case one can use for J the canonical isometry from Y into Y^{**} because Y^{**} is norm one complemented in $(Y^{\#})^*$. Then the proof that $I_p(T) \le I_p^L(T)$ is identical to the proof of Theorem 2.

Problem 2 *Is every Lipschitz 2-summing operator Lipschitz 2-integral?*

In the case where the target space Y is a Hilbert space, problem 2 has an affirmative answer by Kirszbraun's theorem [3, p. 18]. If Y has K. Ball's Markov cotype 2 property [1], it follows from Ball's work that the answer is still positive, although his result does not yield that $I_p^L(T)$ and $\pi_p^L(T)$ are equal. It is worth mentioning that the work of Naor, Peres, Schramm, and Sheffield [9] combines with Ball's result to yield that for $2 \le p < \infty$, every Lipschitz p-summing operator into L_r, $1 < r \le 2$, is Lipschitz p-integral.

We mentioned in section 2 that $\Pi_p^L(X, Y)$, the class of Lipschitz p-summing operators from X into Y, is a Banach space under the norm $\pi_p^L(\cdot)$ when Y is a Banach space.

Problem 3 *When Y is a Banach space and X is finite, what is the dual of $\Pi_p^L(X, Y)$?*

In section 2 we noted that there is a version of Grothendieck's theorem that is true in the non linear setting. Are there other versions? In particular, we ask the following.

Problem 4 *Is every Lipschitz mapping from an L_1 space to a Hilbert space Lipschitz 1-summing? Is every Lipschitz mapping from a $C(K)$ space to a Hilbert space Lipschitz 2-summing?*

It is elementary that for a linear operator $T : X \to Y$, $\pi_p(T)$ is the supremum of $\pi_p(TS)$ as S ranges over all operators from $\ell_{p'}$ into X of norm at most one. This leads us to ask

Problem 5 *If $T : X \to Y$ is Lipschitz, is $\pi_p^L(T)$ is the supremum of $\pi_p^L(TS)$ as S ranges over all mappings from finite subsets of $\ell_{p'}$ into X having Lipschitz constant at most one?*

Since all finite metric spaces embed isometrically into ℓ_∞, the answer to problem 5 is yes for $p = 1$.

Of course, all of the above problems are special cases of the general

Problem 6 *What results about p-summing operators have analogues for Lipschitz p-summing operators?*

References

[1] Ball, K. Markov chains, Riesz transforms and Lipschitz maps. Geom. Funct. Anal. 2 (1992), no. 2, 137–172.

[2] Bates, S.; Johnson, W. B.; Lindenstrauss, J.; Preiss, D.; Schechtman, G. Affine approximation of Lipschitz functions and nonlinear quotients. Geom. Funct. Anal. 9 (1999), no. 6, 1092–1127.

[3] Benyamini, Yoav; Lindenstrauss, Joram. Geometric nonlinear functional analysis. Vol. 1. American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000.

[4] Bourgain, J. On Lipschitz embedding of finite metric spaces in Hilbert space. Israel J. Math. 52 (1985), no. 1-2, 46–52.

[5] Diestel, Joe; Jarchow, Hans; Tonge, Andrew. Absolutely summing operators. Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.

[6] Farmer, Jeffrey D. Extreme points of the unit ball of the space of Lipschitz functions. Proc. Amer. Math. Soc. 121 (1994), no. 3, 807–813.

[7] Johnson, William B.; Lindenstrauss, Joram; Preiss, David; Schechtman, Gideon. Lipschitz quotients from metric trees and from Banach spaces containing l_1. J. Funct. Anal. 194 (2002), no. 2, 332–346.

[8] Johnson, William B.; Schechtman, Gideon.

Diamond graphs and super-reflexivity (submitted).

[9] Naor, Assaf; Peres, Yuval; Schramm, Oded; Sheffield, Scott. Markov chains in smooth Banach spaces and Gromov-hyperbolic metric spaces. Duke Math. J. 134 (2006), no. 1, 165–197. (Reviewer: Keith Ball) 46B09 (46B20 60B11 60J05)

Jeffrey D. Farmer
Department of Mathematics
University of Denver
Denver CO, USA
E-mail: jdfarmer89@hotmail.com

William B. Johnson
Department Mathematics
Texas A&M University
College Station, TX, USA
E-mail: johnson@math.tamu.edu