Non-linear factorization of linear operators *

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Abstract

We show, in particular, that a linear operator between finite dimensional normed spaces, which factors through a third Banach space Z via Lipschitz maps, factors linearly through the identity from $L_\infty([0,1],Z)$ to $L_1([0,1],Z)$ (and thus, in particular, through each $L_p(Z)$, $1 \le p \le \infty$) with the same factorization constant. It follows that, for each $1 \le p \le \infty$, the class of $\mathcal{L}_p$ spaces is closed under uniform (and even coarse) equivalences. The case $p = 1$ is new and solves a problem raised by Heinrich and Mankiewicz in 1982. The proof is based on a simple local-global linearization idea.

1 Introduction

Let X be a pointed metric space with distinguished point, 0. $X^{\#}$ denotes the Banach space of real valued Lipschitz functions f on X for which $f(0) = 0$ under the norm $\mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x)-f(y)|}{\|x-y\|}$. The evaluation map $K_X : X \to X^{\#*}$ is then an isometric isomorphism. If X is a Banach space, then by [5], X^{**} is norm one complemented in $X^{\#*}$ via a projection P that satisfies the identity $PK_X = J_X$, where J_X is the canonical embedding of X into X^{**}.

Given a Lipschitz function $T : X \to Y$ with X and Y pointed metric spaces and $1 \le p \le \infty$, we define the Lipschitz L_p factorization norm $\gamma_p^L(T)$ by

$$\gamma_p^L(T) := \inf\{\mathrm{Lip}(F)\mathrm{Lip}(G) : K_Y \circ T = G \circ F\},$$

where the infimum is over all Lipschitz factorizations of $K_Y \circ T$ through an arbitrary L_p space. By the result of Lindenstrauss mentioned in the previous paragraph, if Y is a normed space this is the same as taking the infimum over all Lipschitz factorizations of $J_Y \circ T$ through an L_p space.

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The main result, Theorem 2, of this paper is that if T is a linear operator between Banach spaces and $1 \leq p \leq \infty$, then $\gamma_p^L(T) = \gamma_p(T)$. (Recall that $\gamma_p(T)$ is defined to be the infimum of $\|F\|\|G\|$ where the infimum is taken over all linear factorizations $J_Y \circ T = G \circ F$ of $J_Y \circ T$ through arbitrary L_p spaces.) The case $p = 1$ of this theorem uses new ideas and gives as a corollary that a Banach space that is uniformly (or even coarsely) equivalent to a $\mathcal{L}_1$ space is again a $\mathcal{L}_1$ space. This answers a question Heinrich and Mankiewicz asked in their 1982 paper [3]. The main result is dealt with in section 3. The proof uses a local-global approach reminiscent of a method recently employed in [4]. Although the proof is fairly simple, easier special cases of Theorem 2 resisted for some time attacks by several experts; for example, it is not completely straightforward to show that if $f_n : \ell_2^n \to L_1$, $g_n : L_1 \to \ell_2^n$ are such that $g_n f_n$ is the identity on ℓ_2^n, then $\mathrm{Lip}(f_n)\mathrm{Lip}(g_n) \to \infty$ as $n \to \infty$, and, indeed, understanding this example led us to Theorem 2.

In section 2 we present another factorization result the proof of which is based on much older and well known ideas but still produces some new information on the invariance of some classes of $\mathcal{L}_1$ spaces under Lipschitz isomorphisms.

Although our research was motivated by problems concerning the classification of Banach spaces under non-linear equivalences, no specialized knowledge of Banach space theory is required for reading this paper. Let us just mention that one equivalence for a Banach space X to be a $\mathcal{L}_p$ space is that the canonical injection $J_X : X \to X^{**}$ factor (linearly) through $L_p(\mu)$ for some measure μ (and, if $1 < p \neq 2 < \infty$, that X not be isomorphic to a Hilbert space); see [6].

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2 The classical argument

Here we prove

Theorem 1 *Assume $T : X \to Y$ is a linear operator from a Banach space X to a dual Banach space Y. Assume T factors through a third Banach space Z via Lipschitz mappings: $T = g \circ f$, $f : X \to Z$, $g : Z \to Y$ and assume in addition that f has a point of Gâteaux differentiability. Then there are linear operators $A : X \to Z$, $B : Z \to Y$ with $T = BA$ and $\|A\|\|B\| \leq \mathrm{Lip}(f)\mathrm{Lip}(g)$.*

Proof: The proof uses a combination of an argument from [3] and a result from [5].

By making suitable translations in the domain and range we may assume that the point of differentiability of f is 0 and that $f(0) = 0$. Define f_n, g_n by $f_n(x) = nf(\frac{x}{n})$, $g_n(z) = ng(\frac{z}{n})$. Since $g \circ f = T$ is linear, we have that $g_n \circ f_n = T$. By hypothesis, $\|f_n(x) - Ax\| \to 0$ for all $x \in X$, for some linear operator $A : X \to Z$. Let $\tilde{g}(x)$ be the weak* limit of $g_n(x)$ through some fixed free ultrafilter of natural numbers. Using the *norm* convergence of $f_n(x)$ to Ax, we deduce that $\tilde{g} \circ A = T$. Of course, $\|A\| \leq \mathrm{Lip}(f)$ and $\mathrm{Lip}(\tilde{g}) \leq \mathrm{Lip}(g)$. So we have linear A and Lipschitz $\tilde{g}$ such that

$$T = \tilde{g} \circ A$$

and $\|A\|\mathrm{Lip}(\tilde{g}) \leq \mathrm{Lip}(f)\mathrm{Lip}(g)$. Also, $\tilde{g}$ is linear on AX, so, by [5], there is a linear operator $B : Z \to Y$ with $\|B\| \leq \mathrm{Lip}(\tilde{g})$ so that $BA = T$. A softer proof of the result we used from [5] is contained in [7]; see also [1, Theorem 7.2]

■

As a corollary we get,

Corollary 1 *Assume $T : X \to Y$ is a linear operator between a separable Banach space X and a Banach space Y. Assume T factors through a third Banach space Z having the Radon–Nikodym property* (RNP) *via Lipschitz mappings: $T = g \circ f$, $f : X \to Z$, $g : Z \to Y$. Then there are linear operators $A : X \to Z$, $B : Z \to Y^{**}$ with $JT = BA$ and $\|A\|\|B\| \leq \mathrm{Lip}(f)\mathrm{Lip}(g)$. Here J is the canonical embedding of Y into its second dual.*

Recall [1, section 6.4] that one equivalence to Z having the RNP is that every Lipschitz mapping from $\mathbb{R}$ (or even from a general separable Banach space) into Z has a point of Gâteaux differentiability. Reflexive spaces and separable conjugate spaces, including ℓ_1, have the RNP, while, for example, c_0 and $L_1[0, 1]$ fail the RNP.

Applying this to the case where $Z = \ell_1$ and to the case where Z is a general separable $\mathcal{L}_1$ space with the Radon–Nikodym property and using the fact that a Banach space X is $\mathcal{L}_1$ whenever the canonical injection of X into its second dual factors through a $\mathcal{L}_1$ space [6], we get

Corollary 2 *The following two families are preserved under Lipschitz isomorphisms:*

1. *The $\mathcal{L}_1$-subspaces of ℓ_1.*

2. *The separable $\mathcal{L}_1$ spaces that have the Radon–Nikodym property.*

3 The local–global argument

Recall that any n-dimensional normed space X admits a normalized basis $x_1, \ldots, x_n$ such that

$$n\|\sum_{i=1}^{n} a_i x_i\| \geq \sum_{i=1}^{n} |a_i| \tag{1}$$

for all $a_1, \ldots, a_n \in \mathbb{R}$. Indeed, any Auerbach basis satisfies this. (See e.g. [1, page 432] for the definition and proof of the existence of an Auerbach basis; alternatively, use any basis for X and replace n on the left side of (1) by a constant which can depend on X and adjust (2) below appropriately.)

The next proposition is the main new tool in the proof of the main Theorem.

Proposition 1 *Let X be an n-dimensional normed space, Y a Banach space and $T : X \to Y$ a bounded linear map. Let $S : X \to Y$ be Gâteaux differentiable everywhere and Lipschitz. Let $x_1, \ldots, x_n$ be a basis for X which satisfies (1), let $\varepsilon > 0$, and assume that*

$$||Sx - Tx\| \leq \varepsilon/2n \tag{2}$$

for all $x \in C = \{\sum_{i=1}^n t_i x_i \; ; \; 0 \leq t_i \leq 1, 1 \leq i \leq n\}$. Then

$$\| \int_C S'(x)\, dx - T\| \leq \varepsilon,$$

where $S'(x)$ is the Gâteaux derivative of S at x and dx is the measure induced on C from the natural map, g, from $[0,1]^n$ onto C and Lebesgue measure on $[0,1]^n$.

Proof: The conclusion follows from inequality (3), which is valid for every Lipschitz function from $[0,1]^n$ into a Banach space which is almost everywhere differentiable (Gâteaux or Fréchet; it is the same for Lipschitz functions from finite dimensional spaces [1, Proposition 4.3]) and every y in $\mathbb{R}^n$:

$$\| \int_{[0,1]^n} f'(t)(y)\, dt\| \leq 2 \max_{t \in \partial[0,1]^n} \|f(t)\| \|y\|_1 \tag{3}$$

(apply (3) to the function $f := (S - T)g$).

To prove (3), let $\{e_i\}_{i=1}^n$ denote the unit vector basis for $\mathbb{R}^n$ and consider the following divergence theorem-like formula

$$\begin{aligned} \int_{[0,1]^n} f'(t)(e_i)\, dt = & \int_0^1 \int_0^1 \ldots \int_0^1 f'(\Sigma_{j \neq i} t_j e_j + s e_i)(e_i)\, ds dt_1 \ldots dt_n \\ = & \int_0^1 \int_0^1 \ldots \int_0^1 f(\sum_{j \neq i} t_j e_j + e_i) - f(\sum_{j \neq i} t_j e_j)\, dt_1 \ldots dt_n. \end{aligned} \tag{4}$$

Multiply (4) by y_i and sum over i to get

$$\begin{aligned} \| \int_{[0,1]^n} f'(t)(y)\, dt\| = & \| \sum_{i=1}^n \int_0^1 \int_0^1 \ldots \int_0^1 y_i (f(\sum_{j \neq i} t_j e_j + e_i) - f(\sum_{j \neq i} t_j e_j))\, dt_1 \ldots dt_n\| \\ \leq & \sum_{i=1}^n \int_0^1 \int_0^1 \ldots \int_0^1 |y_i| (\|f(\sum_{j \neq i} t_j e_j + e_i)\| + \|f(\sum_{j \neq i} t_j e_j)\|)\, dt_1 \ldots dt_n \end{aligned}$$

which clearly implies (3). ■

Theorem 2 *Let X be a finite dimensional normed space, Y a Banach space with the RNP and $T : X \to Y$ a linear operator. Let Z be a separable Banach space and assume there are Lipschitz maps $F_1 : X \to Z$ and $F_2 : Z \to Y$ with $F_2 \circ F_1 = T$. Then for every $\lambda > 1$ there are linear maps $T_1 : X \to L_\infty(Z)$ and $T_2 : L_1(Z) \to Y$ with $T_2 \circ i_{\infty,1} \circ T_1 = T$ and $\|T_1\| \cdot \|T_2\| \leq \lambda \mathrm{Lip}\,(F_1) \mathrm{Lip}(F_2)$.*

Proof: Here $L_\infty(Z)$ and $L_1(Z)$ are the Z-valued measurable functions on $[0,1]$ under the essential sup and L_1 norms, respectively. However, it is equivalent to replace $[0,1]$ with any other separable, purely non atomic measure space, and in fact we use $[0,1]^m$ for a suitable m instead of $[0,1]$. It would of course be better to factor T through Z itself, but it remains open whether this can be done even when Y is also finite dimensional and λ is replaced by any numerical constant.

Set $n = \dim(X)$, let $\{x_i\}_{i=1}^n$ be a basis for X which satisfies (1), and let $\delta > 0$. First note that there are everywhere Gâteaux differentiable maps $\tilde{F}_1 : X \to Z$ and $\tilde{F}_2 : Z \to Y$ with $\mathrm{Lip}(\tilde{F}_i) \le \mathrm{Lip}(F_i)$ and $\|F_i(x) - \tilde{F}_i(x)\| \le \delta$ for all x in the domain of F_i, $i = 1, 2$. The existence of $\tilde{F}_1$ is classical: regard X as $\mathbb{R}^n$ under some norm, let f be a non negative C^1 function supported on a small neighborhood of 0 in X for which $\int_X f(x)\,dx = 1$, and set $\tilde{F}_1 = f * F_1$. The function $\tilde{F}_2$ exists because Y has the RNP; see, for example, [1, Corollary 6.43].

Define $S = \tilde{F}_2 \circ \tilde{F}_1$. Then $\|Sx - Tx\| \le \delta(1 + \mathrm{Lip}(F_2))$ for all x. By Proposition 1,

$$\| \int_C \tilde{F}_2'(\tilde{F}_1(x)) \circ \tilde{F}_1'(x) - T\| = \| \int_C S'(x)\,dx - T\| \le 2n\delta(1 + \mathrm{Lip}(F_2)), \tag{5}$$

where $C = \{\sum_{i=1}^n t_i x_i \ ; \ 0 \le t_i \le 1, 1 \le i \le n\}$. Define now $H_1 : X \to L_\infty([0,1]^n, Z)$ by

$$H_1 x(t) = \tilde{F}_1'(t_1 x_1 + \cdots + t_n x_n)(x)$$

and $H_2 : L_1([0,1]^n, Z) \to Y$ by

$$H_2 f = \int_{[0,1]^n} \tilde{F}_2'(\tilde{F}_1(t_1 x_1 + \cdots + t_n x_n)) f(t)\,dt.$$

Then,

$$\|H_1 u\| \le \sup_{[0,1]^n} \|\tilde{F}_1'(t_1 x_1 + \cdots + t_n x_n)(u)\| \le \mathrm{Lip}(\tilde{F}_1)\|u\|$$

and

$$\|H_2 f\| \le \int_{[0,1]^n} \|\tilde{F}_2'(\tilde{F}_1(t_1 x_1 + \cdots + t_n x_n)) f(t)\|\,dt \le \mathrm{Lip}(\tilde{F}_2) \int_{[0,1]^n} \|f(t)\|\,dt$$

so that

$$\|H_1\| \cdot \|H_2\| \le \mathrm{Lip}(F_1)\mathrm{Lip}(F_2).$$

Also,

$$H_2(H_1 u) = \int_{[0,1]^n} \tilde{F}_2'(\tilde{F}_1(t_1 x_1 + \cdots + t_n x_n))(\tilde{F}_1'(t_1 x_1 + \cdots + t_n x_n)(u))\,dt = \int_C S'(x)(u)\,dx.$$

We thus have found linear $H_1 : X \to L_\infty([0,1]^n, Z)$ and $H_2 : L_1([0,1]^n, Z) \to Y$ with $H := \int_C S'(x)\,dx = H_2 \circ i_{\infty,1} \circ H_1$, $\|H_1\| \cdot \|H_2\| \le \mathrm{Lip}(F_1)\mathrm{Lip}(F_2)$ and, by (5), $\|H - T\| \le 2n\delta(1 + \mathrm{Lip}(F_2))$. This implies that, if δ is small enough, a similar factorization holds for T. To see this, we regard $L_\infty([0,1]^n, Z)$ as being those functions in $L_\infty([0,1]^{2n}, Z)$ which depend only on the first n coordinates and similarly for L_1. Define the operator $\tilde{H}_2 : L_1([0,1]^{2n}, Z) \to Y$ to be the composition of H_2 with the (norm one) conditional expectation projection P from $L_1([0,1]^{2n}, Z)$ onto $L_1([0,1]^n, Z)$. Of course, the kernel of P contains the mean zero Z valued integrable functions on $[0,1]^{2n}$ which depend only on the last n components. In order to get a factorization of $T : X \to Y$ through the injection $L_\infty([0,1]^{2n}, Z) \to L_1([0,1]^{2n}, Z)$ it is enough to show that for all $\epsilon > 0$, if $U : X \to Y$ is a linear operator with sufficiently small norm, then there are operators A from X into the mean zero functions in $L_\infty([0,1]^{2n}, Z)$ which depend only on the last n coordinates and B from $L_1([0,1]^{2n}, Z)$ into Y so that $\|A\|$, $\|B\| < \epsilon$ and $U = Bi_{\infty,1}A$. Indeed, setting $U := T - \tilde{H}_2 H_1$, we see that the pair $H_1 + A$ and $\tilde{H}_2 + B(I - P)$ provides a factorization of T through $L_\infty([0,1]^{2n}, Z) \to L_1([0,1]^{2n}, Z)$, and obviously $\|H_1 + A\| \le \|H_1\| + \epsilon$ and $\|\tilde{H}_2 + B(I - P)\| \le \|H_2\| + 2\epsilon$.

To define A, let z be any norm one vector in Z, let $r := 1_{(0,1/2)} - 1_{(1/2,1)}$ and for $1 \le j \le n$ let r_j be the function on $[0,1]^{2n}$ defined by $r_j(t_1, \dots, t_{2n}) = r(t_{n+j})$. Set $Ax_j = (\epsilon/n) r_j \otimes z$ so that $\|A\| \le \epsilon$. Let Q be a projection of norm at most $\sqrt{n}$ from $L_1([0,1]^{2n}, Z)$ onto the span W of $r_j \otimes z$; $1 \le j \le n$. Define $\tilde{B}$ from W into Y by setting $\tilde{B}(r_j \otimes z) = (n/\epsilon) U x_j$ and define $B := \tilde{B} Q$. Clearly $\|B\| \le (n/\epsilon)\sqrt{n} \max\{\|Ux_j\| : 1 \le j \le n\}$, so we just need $\|U\| < \epsilon^2 n^{-3/2}$.

■

As Corollaries we get

Corollary 3 *Let X and Y be Banach spaces, T a linear operator from X into Y. Then*

$$\gamma_1(T) = \gamma_1^L(T). \tag{6}$$

Proof: To verify (6), it is enough to show for each finite dimensional subspace E of X and each finite codimensional subspace F of Y that

$$\gamma_1(Q_F T i_E) \le \gamma_1^L(T), \tag{7}$$

where i_E is the injection from E into X and Q_F is the quotient mapping from Y onto Y/F, because $\gamma_1(T)$ is the supremum over all such E and F of the left side of (7) (see [2, Theorem 9.1]). Now if $J_Y T = G \circ H$ is a factorization of $J_Y T$ through an L_1 space, then $Q_F^{**} G \circ H i_E$ is a factorization of $Q_F T i_E$ through the same L_1 space, which yields that $\gamma_1^L(Q_F T i_E) \le \gamma_1^L(T)$. Thus it suffices to check that $\gamma_1(Q_F T i_E) \le \gamma_1^L(Q_F T i_E)$; that is, it is enough to verify (6) when X and Y are finite dimensional.

So in the sequel we assume that X and Y are finite dimensional. In this case the desired conclusion follows from Theorem 2. Indeed, if $T = G \circ H$ is a Lipschitz factorization of T through some (possibly non separable) L_1 space Z, then we can replace Z with the (separable) L_1 subspace of Z generated by HX, and then apply Theorem 2. ■

Corollary 4 *Let X be a Banach space and assume that for some L_1 space there are maps $F : X \to L_1$ and $G : L_1 \to X^{**}$ which are Lipschitz for large distances with $G \circ F = I$ (the canonical embedding). Then $\gamma_1(I) < \infty$; i.e., For some L_1 space there are linear $T_1 : X \to L_1$ and $T_2 : L_1 \to X^{**}$ with $T_2 \circ T_1 = I$. Consequently, if X is uniformly or coarsely equivalent to a $\mathcal{L}_1$ space then it is a $\mathcal{L}_1$ space.*

Proof: Before starting the proof we recall that a map F is Lipschitz for large distances provided for all $d > 0$ there is $K_d < \infty$ so that $d(F(x), F(y)) \le K_d d(x, y)$ whenever $d(x, y) \ge d$. Uniformly continuous mappings whose domain is convex (or sufficiently close to being convex) are Lipschitz for large distances [1, Proposition 1.11]. A coarse equivalence between Banach spaces is just a biLipschitz equivalence between a pair of nets in the respective spaces; it is clear that such a map can be extended to a mapping that is Lipschitz for large distances.

We now turn to the proof. A standard ultraproduct argument shows that we may assume that F and G are Lipschitz. (Assume, without loss of generality, that $F(0) = 0$. Let $\mathcal{U}$ be a free ultrafilter on the natural numbers and define $F_n : X \to Y$ by $F_n(x) = \frac{F(nx)}{n}$ and let $\widetilde{F} : X \to (L_1)_{\mathcal{U}}$ be the ultraproduct of the maps F_n into the ultrapower $(L_1)_{\mathcal{U}}$ of L_1, which is another L_1 space. Define $G_n : L_1 \to Y$ by $G_n(z) = G_n(x) = \frac{G(nz)}{n}$ and define $\widetilde{G} : (L_1)_{\mathcal{U}} \to Y^{**}$ by setting $\widetilde{G}(z_1, z_2, \dots) = \mathrm{w}^* - \lim_{n \in \mathcal{U}} G_n(z_n)$. The maps $\widetilde{F}$ and $\widetilde{G}$ are Lipschitz.)

Now use the previous corollary to get that $\gamma_1(I) < \infty$. By [6], X is an $\mathcal{L}_1$ space. ∎

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