Second Examination (Answers)

1. *(36 points)* Let f be a real-valued function defined on the interval $[0,1]$. Do all six of the following problems.

- **(a)** Using sequences, define f is *continuous at* $x_0 \in [0,1]$.
- **(b)** State the ϵ-δ definition of f is *uniformly continuous on* $[0,1]$.
- **(c)** State the *Intermediate Value Theorem* for f. Include all relevant conditions on f.
- **(d)** State the *Mean Value Theorem* for f. Include all relevant conditions on f. (Omitted.)
- **(e)** Define f is *Riemann integrable on* $[0,1]$ in terms of $L_P(f)$ and $U_P(f)$. (Here $L_P(f)$ and $U_P(f)$ denote lower and upper sums associated with a partition P of $[0,1]$. You need not expand these quantities.)
- **(f)** Define the *derivative* $f'(\frac{1}{2})$.

2. *(20 points)* In each of the following two problems, first state whether each given statement is TRUE or FALSE. If the statement is true, then supply a short proof. If the statement is false, then provide a counterexample. In each problem f is a real-valued function.

- **(a)** If f is continuous on $(0,1)$, then f is differentiable on $(0,1)$.

  Answer. FALSE. A counterexample is $f(x) = |x - \frac{1}{2}|$.
- **(b)** If f is continuously differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all x, then f is a constant function. (Omitted.)

3. *(44 points)* Prove each of the following four statements. In these problems f and g are a real-valued functions.

- **(a)** If f is continuous on $[0,1]$, then f is bounded on $[0,1]$.

  Answer. Cf. theorem 3.2.1 in the text.
- **(b)** If f and g are continuous on $\mathbb{R}$, then $f+g$ is continuous. (Prove by ϵ-δ methods.)

  Answer. Cf. theorem 3.1.1 in the text.
- **(c)** If $f(x) = x^2$, then f is differentiable at $x = 3$. (Prove by ϵ-δ methods.)

  Answer. One calclulates that

$$\frac{(x+h)^2 - x^2}{h} = 2x + h.$$

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Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $|h| < \delta$, then

$$\left| \frac{(3+h)^2 - 3^2}{h} - 6 \right| = |h| < \epsilon.$$

(d) Let $f(x) = 3x$ on the interval $[0, 1]$, and let $\epsilon > 0$ be given. Explain explicitly how to construct a partition P so that $U_P(f) - L_P(f) < \epsilon$.

Answer. Let n be a positive integer with $3/n < \epsilon$, and let P be the partition $x_i = i/n$, $i = 0, 1, \ldots , n$. On any subinterval $[x_{i-1}, x_i]$, $m_i = 3x_{i-1}$ and $M_i = 3x_i$ (notation as in §3.3 of the text). Then

$$U_P(f) - L_P(f) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = n(3/n)(1/n) = 3/n < \epsilon.$$

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