

Geometry Summary

Undefined Terms

Our undefined terms are **point**, **line**, **incident**, **betweenness**, and **congruent**.

Incidence Geometry

Axiom (I-1). For every point P and for every point Q not equal to P , there exist a unique line l incident with P and Q .

Axiom (I-2). For every line l , there exist at least two distinct points incident with l .

Axiom (I-3). There exist three distinct points with the property that no line is incident with all three of them.

Definition (Collinear). Three or more points A, B, C, \dots are **collinear** if there exists a point incident with all of them.

Definition (Concurrent). Three or more lines l, m, n, \dots are **concurrent** if there exists a point incident with all of them.

Definition (Parallel). Lines l and m are **parallel** if they are distinct lines and no point is incident with both of them.

Proposition (2.1). If l and m are distinct lines that are not parallel, then l and m have a unique point in common.

Proposition (2.2). There exist three distinct lines that are not concurrent.

Proposition (2.3). For every line, there is at least one point not lying on it.

Proposition (2.4). For every point, there is at least one line not passing through it.

Proposition (2.5). For every point P , there exist at least two lines through P .

Definition (Projective Plane). A **projective plane** is a model of incidence geometry having the **elliptic** parallel property (any two lines meet) and such that every line has at least three distinct points lying on it.

Definition (Affine Plane). An **affine plane** is a model of incidence geometry having the following **Euclidean** parallel property: for every line l and every point P not on l , there exists a unique line m such that P is incident to m and m is parallel to l .

Betweenness

Axiom (B-1). If $A * B * C$, then A , B , and C are three distinct points all lying on the same line, and $C * B * A$.

Axiom (B-2). Given any two distinct points B and D , there exist points A , C , and E lying on \overleftrightarrow{BD} such that $A * B * D$, $B * C * D$, and $B * D * E$.

Axiom (B-3). If A , B , and C are three distinct points lying on the same line, then one and only one of the points is between the other two.

Proposition (3.1). For any two points A and B :

(i) $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$.

(ii) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

Definitions (segment, Ray).

(1) The **segment** AB is the set of all points between A and B together with the endpoints A and B .

(2) The **ray** \overrightarrow{AB} is the set of all points on the segment AB together with all the points C such that $A * B * C$.

Definition (Same and Opposite Sides). Let l be any line, and let A and B be any points that do not lie on l . If $A = B$ or the segment AB has no point lying on l , then we say that A and B are **on the same side of** l . If $A \neq B$ and segment AB does intersect l , then we say that A and B are **on opposite sides of** l .

Axiom (B-4 Plane Separation). For every line l and for any three points A , B , and C not lying on l :

(i) If A and B are on the same side of l and if B and C are on the same side of l , then A and C are on the same side of l .

(ii) If A and B are on opposite sides of l and if B and C are on opposite sides of l , then A and C are on the same side of l .

Corollary (iii). If A and B are on opposite sides of l and if B and C are on the same side of l , then A and C are on opposite sides of l .

Definitions (Side, Half-Plane). A **side** of a line l is the set of all points that are on the same side of l as some particular point A not on l . This side is denoted H_A . Another expression for a side of a line l is a **half-plane bounded by l** .

Proposition (3.2). Every line bounds exactly two half-planes, and those half-planes have no point in common.

Proposition (3.3). Given $A * B * C$ and $A * C * D$, then $B * C * D$ and $A * B * D$.

Corollary. Given $A * B * C$ and $B * C * D$, then $A * B * D$ and $A * C * D$.

Proposition (3.4). If $C * A * B$ and l is the line through A , B , and C , then for every point P lying on l , P lies either on ray \overrightarrow{AB} or on the opposite ray \overrightarrow{AC} .

Theorem (Pasch's Theorem). If A , B , C are distinct noncollinear points and l is a line intersecting AB in a point between A and B , then l also intersects either AC or BC . If C does not lie on l , then l does not intersect both AC and BC .

Proposition (3.5). Given $A * B * C$, then $AC = AB \cup BC$ and B is the only point common to segments AB and BC .

Proposition (3.6). Given $A * B * C$, then B is the only point common to rays \overrightarrow{AB} and \overrightarrow{BC} , and $\overrightarrow{AB} = \overrightarrow{AC}$.

Definition (Interior of an Angle). Given $\angle CAB$, a point D is in the **interior** of $\angle CAB$ if D is on the same side of \overleftrightarrow{AC} as B and if D is also on the same side of \overleftrightarrow{AB} as C .

Proposition (3.7). Given an angle $\angle CAB$ and point D lying on line \overleftrightarrow{BC} , then D is in the interior of $\angle CAB$ if and only if $B * D * C$.

Proposition (3.8). If D is on the interior of $\angle CAB$, then:

- (a) so is every other point on ray \overrightarrow{AD} except A ;
- (b) no point on the opposite ray to \overrightarrow{AD} is in the interior of $\angle CAB$; and
- (c) if $C * A * E$, then B is in the interior of $\angle DAE$.

Definition (Ray Between Two Rays). Ray \overrightarrow{AD} is **between** rays \overrightarrow{AC} and \overrightarrow{AB} if \overrightarrow{AC} and \overrightarrow{AB} are not opposite rays and D is interior to $\angle CAB$.

Theorem (Crossbar Theorem). If \overrightarrow{AD} is between \overrightarrow{AC} and \overrightarrow{AB} , then \overrightarrow{AD} intersects segment BC .

Definitions (Interior and Exterior of a Triangle). The **interior of a triangle** is the intersection of the interiors of its three angles. A point is **exterior to the triangle** if it is not in the interior of the triangle and does not lie on any side of the triangle

Proposition (3.9).

- (a) If a ray r emanating from an exterior point of $\triangle ABC$ intersects side AB in a point between A and B , then r also intersects side AC or side BC .
- (b) If a ray emanates from an interior point of $\triangle ABC$, then it intersects one of the sides, and if does not pass through a vertex, it intersects only one side.

Congruence

Definition (Congruent Triangles). Triangles $\triangle ABC$ and $\triangle DEF$ are **congruent** if there exists a one-to-one correspondence between their vertices such that corresponding sides are congruent and corresponding angles are congruent.

Note that the notation $\triangle ABC \cong \triangle DEF$ indicates not only that these triangles are congruent, but that there is a correspondence demonstrating that congruence such that A corresponds to D , B to E , and C to F .

Axiom (C-1). If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a unique point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.

Axiom (C-2). If $AB \cong CD$ and if $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

Axiom (C-3). If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

Axiom (C-4). Given any $\sphericalangle BAC$ and given any ray $\overrightarrow{A'B'}$ emanating from point A' , then there is a unique ray $\overrightarrow{A'C'}$ on a given side of $\overrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$.

Axiom (C-5). If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

Axiom (C-6 SAS). If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then the two triangles are congruent.

Corollary (Corollary to SAS). Given $\triangle ABC$ and segment $DE \cong AB$, there is a unique point F on a given side of line \overleftrightarrow{DE} such that $\triangle ABC \cong \triangle DEF$.

Proposition (3.10 Isosceles Triangles). If in $\triangle ABC$ we have $AB \cong AC$, then $\sphericalangle B \cong \sphericalangle C$.

Proposition (3.11 Segment Subtraction). If $A * B * C$, $D * E * F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.

Proposition (3.12). Given $AC \cong DF$, then for any point B between A and C , there is a unique point E between D and F such that $AB \cong DE$.

Definition (Segment Ordering). $AB < CD$ (or $CD > AB$) means that there exists a point E between C and D such that $AB \cong CE$.

Proposition (3.13 Segment Ordering).

- (a) (**Trichotomy**) Exactly one of the following three conditions holds: $AB < CD$, $AB \cong CD$, or $AB > CD$.
- (b) If $AB < CD$ and $CD \cong EF$, then $AB < EF$.
- (c) If $AB > CD$ and $CD \cong EF$, then $AB > EF$.
- (d) (**Transitivity**) If $AB < CD$ and $CD < EF$, then $AB < EF$.

Proposition (3.14). Supplements of congruent angles are congruent.

Proposition (3.15).

- (a) Vertical angles are congruent to each other.
- (b) An angle congruent to a right angle is a right angle.

Proposition (3.16). For every line l and every point P , there exists a line through P perpendicular to l .

Proposition (3.17 ASA Criterion for Congruence). If $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A = \sphericalangle D$, $\sphericalangle C = \sphericalangle F$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

Proposition (3.18 Converse of Isosceles Triangles Proposition 3.10). If in $\triangle ABC$, we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

Proposition (3.19 Angle Addition). Give \overrightarrow{BG} between \overrightarrow{BA} and \overrightarrow{BC} , \overrightarrow{EH} between \overrightarrow{ED} and \overrightarrow{EF} , $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle GBA \cong \sphericalangle HED$, then $\sphericalangle ABC \cong \sphericalangle DEF$.

Proposition (3.20 Angle Subtraction). Given \overrightarrow{BG} between \overrightarrow{BA} and \overrightarrow{BC} , \overrightarrow{EH} between \overrightarrow{ED} and \overrightarrow{EF} , $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle ABC \cong \sphericalangle DEF$, then $\sphericalangle GBA \cong \sphericalangle HED$.

Definition (Ordering of Angles). $\sphericalangle ABC < \sphericalangle DEF$ means there is a ray \overrightarrow{EG} between \overrightarrow{ED} and \overrightarrow{EF} such that $\sphericalangle ABC \cong \sphericalangle GEF$.

Proposition (3.21 Ordering of Angles).

- (a) (**Trichotomy**) Exactly one of the following three conditions holds: $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle P > \sphericalangle Q$.
- (b) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
- (c) If $\sphericalangle P > \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P > \sphericalangle R$.
- (d) (**Transitivity**) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

Proposition (3.22 SSS Criterion for Congruence). If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

Proposition (3.23 Euclid's Fourth Postulate). All right angles are congruent to each other.

Definitions (Acute and Obtuse Angles). An angle is **acute** if it is less than a right angle, and **obtuse** if it is greater than a right angle.

Definition (Hilbert Plane). A model of the incidence, betweenness, and congruence axioms is called a **Hilbert plane**.

Axioms of Continuity

Definitions (Circle, Center, Radius). Given distinct points O and A , the set of all points P such that the segment OP is congruent to the segment OA is called the **circle of center O and radius OA** .

Principle (Circle-Circle Continuity Principle). If a circle γ has one point inside and one point outside another circle γ' , then the two circles intersect in two points.

Principle (Line-Circle Continuity Principle). If a line passes through a point inside a circle, then the line intersects the circle in two points.

Principle (Segment-Circle Continuity Principle). If one endpoint of a segment is inside a circle and the other is outside, then the segment intersects the circle at a point in between.

Principle (Archimedes' Axiom). If CD is any segment, A any point, and r any ray with vertex A , then for every point $B \neq A$ on r there is a number n such that when CD is laid off n times on r starting at A , a point E is reached such that $n \cdot CD \cong AE$ and either $B = E$ or B is between A and E .

Principle (Aristotle's Angle Unboundedness Axiom). Given any side of an acute angle and any segment AB , there exists a point Y on the given side of the angle such that if X is the foot of the perpendicular from Y to the other side of the angle, $XY > AB$.

Corollary (Corollary to Aristotle's Axiom). Let \overrightarrow{AB} be any ray, P any point not collinear with A and B , and $\sphericalangle XVY$ any acute angle. Then there exists a point R on ray \overrightarrow{AB} such that $\sphericalangle PRA < \sphericalangle XVY$.

Principle (Dedekind's Axiom). Suppose that the set $\{l\}$ of all points on a line l is the disjoint union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of either subset is between two points of the other. Then there exists a unique point D on l such that one of the subsets is equal to a ray of l with vertex O and the other subset is equal to the complement.

Hilbert's Euclidean Axiom of Parallelism

Principle (Hilbert's Axiom of Parallelism). For every line l and every point P not lying on l , there is at most one line m through P such that m is parallel to l .

Definition (Euclidean Plane). A **Euclidean plane** is a Hilbert plane in which the circle-circle continuity principle and Hilbert's Euclidean axiom of parallelism hold.

Alternate Angle Theorem

Theorem (4.1 Alternate Interior Angle (AIA) Theorem). In any Hilbert plane, if two lines cut by a transversal have a pair of congruent alternate interior angles with respect to that transversal, then the two lines are parallel.

Corollary (Corollary 1). Two lines perpendicular to the same line are parallel. hence the perpendicular dropped from a point P not on line l to l is unique. (The point at which the perpendicular intersects l is called its **foot**.)

Corollary (Corollary 2). If l is any line and P is any point not on l , there exists at least one line m through P parallel to l . (This is Euclid I.31.)

Exterior Angle Theorem

Theorem (Exterior Angle (EA) Theorem 4.2). In any Hilbert plane, an exterior angle of a triangle is greater than either remote interior angle.

Corollary (Corollary 1). If a triangle has a right or obtuse angle, the other two angles are acute.

Proposition (4.1 SAA Congruence Criterion). Given $AC \cong DF$, $\sphericalangle A \cong \sphericalangle D$, and $\sphericalangle B \cong \sphericalangle E$, then $\triangle ABC \cong \triangle DEF$.

Proposition (4.2 Hypotenuse-Leg Criterion). Two right triangles are congruent if the hypotenuse and a leg of one are congruent, respectively, to the hypotenuse and a leg of the other.

Proposition (4.3 Midpoints). Every segment has a unique midpoint.

Proposition (4.4 Bisectors).

- (a) Every angle has a unique bisector.
- (b) Every segment has a unique perpendicular bisector.

Proposition (4.5). In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle; i.e., $AB > AC$ if and only if $\sphericalangle C > \sphericalangle A$.

Proposition (4.6). Given $\triangle ABC$ and $\triangle A'B'C'$, if we have $AB \cong A'B'$ and $BC \cong B'C'$, then $\sphericalangle B > \sphericalangle B'$ if and only if $AC < A'C'$.