Notes on Curvilinear Coordinates*

Jay R. Walton

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1 Introduction

These notes contain a brief introduction to working with curvilinear coordinates in $\mathbb{R}^N$. The vector notation $\mathbf{x} = (x^1, \ldots, x^N)^T$ is used to denote a *Curvilinear Coordinate System* on a region $\mathcal{D} \subset \mathbb{R}^N$ which is defined through a one-to-one, smooth, mapping $\mathbf{z} = \hat{\mathbf{z}}(\mathbf{x})$

$$\mathbf{z} = \hat{\mathbf{z}}(\cdot) : \ \mathcal{B} \subset \mathbb{R}^N \longrightarrow \mathcal{D} \subset \mathbb{R}^N. \tag{1}$$

The function $\hat{\mathbf{z}}(\mathbf{x})$ assigns the position $\mathbf{z} \in \mathcal{D}$ of the point $\mathbf{z}$ with curvilinear coordinate $\mathbf{x}$. The inverse mapping to the assignment function is denoted by $\hat{\mathbf{x}}(\mathbf{z})$. By definition, it must satisfy

$$\hat{\mathbf{x}}(\hat{\mathbf{z}}(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad \hat{\mathbf{z}}(\hat{\mathbf{x}}(\mathbf{z})) = \mathbf{z}. \tag{2}$$

To each point $\mathbf{z}$ with curviliner coordinate $\mathbf{x}$, one constructs a *Local Basis* consisting of the vectors $\{\mathbf{g}_1, \ldots, \mathbf{g}_N\}$ defined through

$$\mathbf{g}_k := \partial_{x^k} \hat{\mathbf{z}}(\mathbf{x}) = \mathbf{z}_{,k}(\mathbf{x}). \tag{3}$$

It is natural to assume

$$0 < \det[\mathrm{D}_{\mathbf{x}} \hat{\mathbf{z}}(\mathbf{x})]. \tag{4}$$

Let $\{\mathbf{g}^1, \ldots, \mathbf{g}^N\}$ denote the reciprocal basis (dual basis) to $\{\mathbf{g}_1, \ldots, \mathbf{g}_N\}$. It follows from (**??**) that

$$\mathbf{g}^k(\mathbf{x}) = \nabla_{\mathbf{z}} \hat{x}^k(\hat{\mathbf{z}}(\mathbf{x})). \tag{5}$$

To see this, consider the following calculation.

$$\begin{aligned} I &= \mathrm{D}_{\mathbf{x}}(\hat{\mathbf{x}}(\hat{\mathbf{z}}(\mathbf{x}))) \\ &= \mathrm{D}_{\mathbf{z}} \hat{\mathbf{x}}(\hat{\mathbf{z}}(\mathbf{x})) \mathrm{D}_{\mathbf{x}} \hat{\mathbf{z}}(\mathbf{x}) \quad \text{by the Chain Rule} \\ &= \left[\nabla_{\mathbf{z}} \hat{x}^j(\mathbf{z}) \cdot \partial_{x^k} \hat{\mathbf{z}}(\mathbf{x}) \right] \quad \text{by the definition of matrix multiplication} \\ &= \left[\nabla_{\mathbf{z}} \hat{x}^j(\mathbf{z}) \cdot \mathbf{g}_k(\mathbf{x}) \right]. \end{aligned}$$

2 Covariant and Contravariant Coordinates

A vector field $\mathbf{w}(\mathbf{x})$ then has both a contravariant and covariant coordinate representation. Specifically,

$$\mathbf{w}(\mathbf{x}) = w^k(\mathbf{x})\mathbf{g}_k(\mathbf{x}) = w_k(\mathbf{x})\mathbf{g}^k(\mathbf{x})$$

with contravariant coordinates $[w^k(\mathbf{x})]$ and covariant coordinates $[w_k(\mathbf{x})]$.

Similarly, a second order tensor field $T(\mathbf{x}) \in \mathcal{T}^2$ has the multiple component representations

$$T(\mathbf{x}) = T^{kl}(\mathbf{x})\mathbf{g}_k(\mathbf{x}) \otimes \mathbf{g}_l(\mathbf{x}) = T_l^k(\mathbf{x})\mathbf{g}_k(\mathbf{x}) \otimes \mathbf{g}^l(\mathbf{x}) = T_{kl}(\mathbf{x})\mathbf{g}^k(\mathbf{x}) \otimes \mathbf{g}^l(\mathbf{x}).$$

$[T^{kl}]$ gives the pure contravariant representation of T, $[T_{kl}]$ the pure covariant representation and $[T_l^k]$ the mixed representation.

Example. The identity tensor $I \in \mathcal{T}^2$ has the following representations

$$I = g^{kl}(\mathbf{x})\mathbf{g}_k(\mathbf{x}) \otimes \mathbf{g}_l(\mathbf{x}) = g_l^k(\mathbf{x})\mathbf{g}_k(\mathbf{x}) \otimes \mathbf{g}^l(\mathbf{x}) = g_{kl}(\mathbf{x})\mathbf{g}^k(\mathbf{x}) \otimes \mathbf{g}^l(\mathbf{x})$$

with

$$g^{kl} = \mathbf{g}^k \cdot \mathbf{g}^l, \quad g_l^k = \mathbf{g}^k \cdot \mathbf{g}_l = \delta_l^k \quad \text{and} \quad g_{kl} = \mathbf{g}_k \cdot \mathbf{g}_l. \tag{6}$$

The tensors $[g^{kl}]$, $[g_l^k]$ and $[g_{kl}]$ are called the contravariant, mixed and covariant components of the *Metric Tensor* I, respectively. Thus, one sees that unless the local basis $\{\mathbf{g}_1, \ldots, \mathbf{g}_N\}$ is an orthonormal basis, only the mixed component matrix representation for the identity tensor has the customary form. The reader should verify the following useful identities.

$$\mathbf{g}^k = g^{kl}\mathbf{g}_l \quad \text{and} \quad \mathbf{g}_k = g_{kl}\mathbf{g}^l.$$

Therefore, the matrices $[g^{kl}]$ and $[g_{kl}]$ can be used to transform between covariant and contravariant component representations of tensors of all orders.

For example, let $\mathbf{w}(\mathbf{x})$ be a vector field defined on $\mathcal{D}$. Then,

$$\mathbf{w}(\mathbf{x}) = w^k(\mathbf{x})\mathbf{g}_k(\mathbf{x}) = w^k g_{kl}\mathbf{g}^l = w_l(\mathbf{x})\mathbf{g}^l(\mathbf{x})$$

with

$$w_l(\mathbf{x}) = w^k(\mathbf{x})g_{kl}(\mathbf{x}).$$

Similarly, one sees that

$$w^l(\mathbf{x}) = w_k(\mathbf{x})g^{kl}(\mathbf{x}).$$

Examples. Let $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ denote the natural orthonormal basis for $\mathbb{R}^3$.

Polar Coordinates. The *Polar Coordinate* system for $\mathbb{R}^2$ is defined by

$$\mathbf{x} = (x^1, x^2) = (r, \theta) \quad \text{with} \quad \hat{\mathbf{z}} = r\cos(\theta)\mathbf{i}_1 + r\sin(\theta)\mathbf{i}_2.$$

The local basis becomes $\{\mathbf{g}_1, \mathbf{g}_2\}$ with

$$\mathbf{g}_1(\theta) = \cos(\theta)\mathbf{i}_1 + \sin(\theta)\mathbf{i}_2 \quad \text{and} \quad \mathbf{g}_2 = -r\sin(\theta)\mathbf{i}_1 + r\cos(\theta)\mathbf{i}_2.$$

For this example, it is useful to introduce the local orthonormal basis $\{\mathbf{j}_1(\theta), \mathbf{j}_2(\theta)\}$

$$\mathbf{j}_1(\theta) := \cos(\theta)\mathbf{i}_1 + \sin(\theta)\mathbf{i}_2 \quad \text{and} \quad \mathbf{j}_2(\theta) := -\sin(\theta)\mathbf{i}_1 + \cos(\theta)\mathbf{i}_2. \tag{7}$$

With this notation, one then has

$$\mathbf{g}_1(\theta) = \mathbf{j}_1(\theta) \quad \text{and} \quad \mathbf{g}_2(r, \theta) = r\mathbf{j}_2(\theta).$$

The reader should now construct the local reciprocal basis.

Cylindrical Coordinates. The *Cylindrical Coordinate* system for $\mathbb{R}^3$ is defined by

$$\mathbf{x} = (x^1, x^2, x^3) = (r, \theta, z) \quad \text{with} \quad \hat{\mathbf{z}} = r\cos(\theta)\mathbf{i}_1 + r\sin(\theta)\mathbf{i}_2 + z\mathbf{i}_3.$$

The local basis becomes

$$\mathbf{g}_1(\theta) = \cos(\theta)\mathbf{i}_1 + \sin(\theta)\mathbf{i}_2, \quad \mathbf{g}_2(r, \theta) = -r\sin(\theta)\mathbf{i}_1 + r\cos(\theta)\mathbf{i}_2 \quad \text{and} \quad \mathbf{g}_3 = \mathbf{i}_3. \tag{8}$$

Again it proves useful to introduce the local orthonormal basis $\{\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3\}$ with $\mathbf{j}_1$ and $\mathbf{j}_2$ as in (**??**) and $\mathbf{j}_3 := \mathbf{i}_3$. One then has

$$\mathbf{g}_1(\theta) = \mathbf{j}_1(\theta), \quad \mathbf{g}_2(r, \theta) = r\mathbf{j}_2(\theta) \quad \text{and} \quad \mathbf{g}_3 = \mathbf{j}_3. \tag{9}$$

The reader should again construct the corresponding reciprocal basis.

Spherical Coordinates. The *Spherical Coordinate* system for $\mathbb{R}^3$ is defined by

$$\mathbf{x} = (x^1, x^2, x^3) = (\rho, \theta, \phi) \quad \text{with} \quad \hat{\mathbf{z}} = \rho\cos(\theta)\sin(\phi)\mathbf{i}_1 + \rho\sin(\theta)\sin(\phi)\mathbf{i}_2 + \rho\cos(\phi)\mathbf{i}_3.$$

The local basis becomes

$$\begin{aligned}
\mathbf{g}_1(\theta, \phi) &= \cos(\theta)\sin(\phi)\mathbf{i}_1 + \sin(\theta)\sin(\phi)\mathbf{i}_2 + \cos(\phi)\mathbf{i}_3 \\
\mathbf{g}_2(\rho, \theta) &= -\rho\sin(\theta)\sin(\phi)\mathbf{i}_1 + \rho\cos(\theta)\sin(\phi)\mathbf{i}_2 \\
\mathbf{g}_3(\rho, \theta, \phi) &= \rho\cos(\theta)\cos(\phi)\mathbf{i}_1 + \rho\sin(\theta)\cos(\phi)\mathbf{i}_2 - \rho\sin(\phi)\mathbf{i}_3.
\end{aligned} \tag{10}$$

There is also a useful local orthonormal basis for the spherical coordinate system given by

$$\begin{aligned}
\mathbf{k}_1(\theta, \phi) &:= \cos(\theta)\sin(\phi)\mathbf{i}_1 + \sin(\theta)\sin(\phi)\mathbf{i}_2 + \cos(\phi)\mathbf{i}_3 \\
\mathbf{k}_2(\theta) &:= -\sin(\theta)\mathbf{i}_1 + \cos(\theta)\mathbf{i}_2 \\
\mathbf{k}_3(\theta, \phi) &:= \cos(\theta)\cos(\phi)\mathbf{i}_1 + \sin(\theta)\cos(\phi)\mathbf{i}_2 - \sin(\phi)\mathbf{i}_3
\end{aligned} \tag{11}$$

with respect to which

$$\mathbf{g}_1(\theta, \phi) = \mathbf{k}_1(\theta, \phi), \quad \mathbf{g}_2(\rho, \theta, \phi) = \rho\sin(\phi)\mathbf{k}_2(\theta) \quad \text{and} \quad \mathbf{g}_3(\rho, \theta, \phi) = \rho\mathbf{k}_3(\theta, \phi). \tag{12}$$

The reader should verify that the local basis (**??**) is indeed orthonormal and construct from (**??**) the reciprocal base vectors $\{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$.

3 Differential Operators

This section derives component formulas for various differential operators applied to tensors represented through a curvilinear coordinate system. First one considers the partial differential operator ∂_{x^k}. For any tensor $\Phi(\mathbf{x})$, it is useful to employ the subscript notation for partial differentiation $\partial_{x^m}\Phi(\mathbf{x}) = \Phi_{,m}$ for $m = 1, \ldots, N$.

Consider first a vector field with covariant and contravariant forms

$$\mathbf{w} = w^k \mathbf{g}_k = w_k \mathbf{g}^k.$$

One then has for the contravariant form

$$\mathbf{w}_{,m} = w^k_{,m}\mathbf{g}_k + w^k \mathbf{g}_{k,m}. \tag{13}$$

It is now necessary to express the vector $\mathbf{g}_{k,m}$ in covariant and contravariant component forms. To that end, one has

$$\mathbf{g}_{k,m} = \Gamma^p_{km}\mathbf{g}_p = \Gamma_{kmp}\mathbf{g}^p \tag{14}$$

where the *Christoffel Symbols* Γ^p_{km} and Γ_{kmp} are given by

$$\Gamma^p_{km} = \mathbf{g}_{k,m} \cdot \mathbf{g}^p \quad \text{and} \quad \Gamma_{kmp} = \mathbf{g}_{k,m} \cdot \mathbf{g}_p.$$

Substitution of (**??**) into (**??**) yields

$$\mathbf{w}_{,m} = w^k_{|m}\mathbf{g}_k$$

where the *Covariant Derivative* $w^k_{|m}$ of the contravariant components of $\mathbf{w}$ is defined by

$$w^k_{|m} := w^k_{,m} + w^p \Gamma^k_{pm}. \tag{15}$$

For the covariant form of $\mathbf{w}$, one has

$$\mathbf{w}_{,m} = w_{k,m}\mathbf{g}^k + w_k \mathbf{g}^k_{,m}. \tag{16}$$

Next one writes

$$\mathbf{g}^k_{,m} = (\mathbf{g}^k_{,m} \cdot \mathbf{g}_p)\mathbf{g}^p \tag{17}$$

and then shows that

$$\mathbf{g}^p_{,m} \cdot \mathbf{g}_k = -g^{rs} g_{rk} \Gamma^p_{sm} = -\delta^s_k \Gamma^p_{sm} = -\Gamma^p_{km}. \tag{18}$$

Substitution of (**??**,**??**) into (**??**) yields

$$\mathbf{w}_{,m} = w_{k,m}\mathbf{g}^k - w_p \Gamma^p_{km}\mathbf{g}^k = w_{k|m}\mathbf{g}^k \tag{19}$$

where the *Covariant Derivative* $w_{p|m}$ of the covariant components of $\mathbf{w}$ is defined by

$$w_{k|m} := w_{k,m} - w_p \Gamma^p_{km}. \tag{20}$$

The derivation of (??) follows from the formula (which the reader should verify)

$$g^{pk}_{,l}(\mathbf{x}) = -g^{pr}(\mathbf{x})\Gamma^k_{rl}(\mathbf{x}) - g^{kr}(\mathbf{x})\Gamma^p_{rl}(\mathbf{x}) \tag{21}$$

and the observation that

$$\begin{aligned}
\mathbf{g}^p_{,m}\cdot\mathbf{g}_k &= (g^{pr}\mathbf{g}_r)_{,m}\cdot\mathbf{g}_k \\
&= (g^{pr}_{,m}\mathbf{g}_r + g^{pr}\mathbf{g}_{r,m})\cdot\mathbf{g}_k \\
&= g^{pr}_{,m}g_{rk} + g^{pr}g_{sk}\Gamma^s_{rm} \\
&= -g^{ps}g_{rk}\Gamma^r_{sm} - g^{rs}g_{rk}\Gamma^p_{sm} + g^{pr}g_{sk}\Gamma^s_{rm} \\
&= -g^{rs}g_{rk}\Gamma^p_{sm} = -\delta^s_k\Gamma^p_{sm} = -\Gamma^p_{km}.
\end{aligned}$$

In similar fashion, one can derive formulae for partial differentiation of second order tensors. For example, given the pure contravariant representation of the second order tensor field $T(\mathbf{x})$

$$T = T^{kl}\mathbf{g}_k\otimes\mathbf{g}_l,$$

one can show that

$$\begin{aligned}
T_{,m} &= T^{kl}_{,m}\mathbf{g}_k\otimes\mathbf{g}_l + T^{kl}\mathbf{g}_{k,m}\otimes\mathbf{g}_l + T^{kl}\mathbf{g}_k\otimes\mathbf{g}_{l,m} \\
&= T^{pq}_{|m}\mathbf{g}_p\otimes\mathbf{g}_q
\end{aligned}$$

where the *Covariant Derivative* $T^{pq}_{|m}$ of the contravariant components of the second order tensor T is defined by

$$T^{pq}_{|m} := T^{pq}_{,m} + T^{jq}\Gamma^p_{jm} + T^{pj}\Gamma^q_{jm}.$$

Analogous formulae can be derived for pure covariant and mixed representations of the second order tensor T.

Gradient. To compute the gradient operator with respect to the curvilinear coordinate formulation given previously, one can proceed as follows. Let $\phi(\mathbf{z}(\mathbf{x})) = \tilde{\phi}(\mathbf{x})$ be a scalar field. Then one has

$$\begin{aligned}
\nabla_{\mathbf{z}}\phi &= \partial_{z^k}\phi\mathbf{e}_k \\
&= \partial_{x^j}\tilde{\phi}(\mathbf{x})\partial_{z^k}\tilde{x}^j\mathbf{e}_k \\
&= \partial_{x^j}\tilde{\phi}(\mathbf{x})\mathbf{g}^j = \tilde{\phi}_{,j}(\mathbf{x})\mathbf{g}^j(\mathbf{x})
\end{aligned}$$

where $\{\mathbf{e}^1,\ldots,\mathbf{e}^N\}$ is the natural orthonormal basis for $\mathbb{R}^N$. Therefore, one sees that for scalar fields the gradient in curvilinear coordinates becomes

$$\nabla = \mathbf{g}^j(\mathbf{x})\partial_{x^j}. \tag{22}$$

To compute the gradient of a vector field, it proves useful to consider first the special cases $\nabla\mathbf{g}_k(\mathbf{x})$ and $\nabla\mathbf{g}^k(\mathbf{x})$. For the former case, one shows that

$$\begin{aligned}
\nabla\mathbf{g}_k(\mathbf{x}) &= \partial_{z^j}\mathbf{g}_k(\mathbf{x})\otimes\mathbf{e}_j \\
&= (\mathbf{g}_{k,m}(\mathbf{x})\partial_{z^j}\tilde{x}^m)\otimes\mathbf{e}_j \\
&= \mathbf{g}_{k,m}(\mathbf{x})\otimes(\partial_{z^j}\tilde{x}^m\mathbf{e}_j) \\
&= \Gamma^p_{km}(\mathbf{x})\mathbf{g}_p(\mathbf{x})\otimes\mathbf{g}^m(\mathbf{x}).
\end{aligned} \tag{23}$$

Similarly, one computes

$$\begin{aligned}
\nabla \mathbf{g}^k(\mathbf{x}) &= \nabla \left(g^{kp}(\mathbf{x}) \mathbf{g}_p(\mathbf{x}) \right) \\
&= g^{kp}(\mathbf{x}) \nabla \mathbf{g}_p(\mathbf{x}) + \mathbf{g}_p(\mathbf{x}) \otimes \nabla g^{kp}(\mathbf{x}) \\
&= g^{kr}(\mathbf{x}) \Gamma^p_{rm}(\mathbf{x}) \mathbf{g}_p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) + \mathbf{g}_p(\mathbf{x}) \otimes (g^{kp}_{,m}(\mathbf{x}) \mathbf{g}^m(\mathbf{x})) \\
&= -g^{pr}(\mathbf{x}) \Gamma^k_{rm}(\mathbf{x}) \mathbf{g}_p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) = -\Gamma^k_{pm}(\mathbf{x}) \mathbf{g}^p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}). \qquad (24)
\end{aligned}$$

For a vector field expressed in terms of its contravariant coordinate functions, $\mathbf{v}(\tilde{\mathbf{z}}(\mathbf{x})) = \tilde{\mathbf{v}}(\mathbf{x}) = \tilde{v}^k(\mathbf{x}) \mathbf{g}_k(\mathbf{x})$, one computes the gradient as follows.

$$\begin{aligned}
\nabla \mathbf{v}(\mathbf{x}) &= \mathbf{g}_k \otimes \nabla \tilde{v}^k(\mathbf{x}) + \tilde{v}^k(\mathbf{x}) \nabla \mathbf{g}_k(\mathbf{x}) \\
&= \mathbf{g}_k(\mathbf{x}) \otimes \mathbf{g}^j(\mathbf{x}) \tilde{v}^k_{,j}(\mathbf{x}) + \tilde{v}^k(\mathbf{x}) \Gamma^p_{km}(\mathbf{x}) \mathbf{g}_p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) \\
&= \left(\tilde{v}^p_{,m}(\mathbf{x}) + \tilde{v}^k(\mathbf{x}) \Gamma^p_{km}(\mathbf{x}) \right) \mathbf{g}_p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) \qquad (25) \\
&= \tilde{v}^p_{|m} \, \mathbf{g}_p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) \qquad (26) \\
&= (\partial_{x^j} \tilde{\mathbf{v}}(\mathbf{x})) \otimes \mathbf{g}^j(\mathbf{x}). \qquad (27)
\end{aligned}$$

Thus, ones sees from (**??**) that the gradient of a vector field has the customary component form with covariant differentiation of contravariant components replacing ordinary partial differentiation.

On the other hand, if the vector field is expressed with respect to its covariant coordinates, $\mathbf{v}(\tilde{\mathbf{z}}(\mathbf{x})) = \tilde{\mathbf{v}}(\mathbf{x}) = \tilde{v}_k(\mathbf{x}) \mathbf{g}^k(\mathbf{x})$, the gradient is conveniently calculated as follows.

$$\begin{aligned}
\nabla \mathbf{v}(\mathbf{x}) &= \tilde{v}_k(\mathbf{x}) \nabla \mathbf{g}^k(\mathbf{x}) + \mathbf{g}^k(\mathbf{x}) \otimes \nabla \tilde{v}_k(\mathbf{x}) \\
&= -\tilde{v}_k(\mathbf{x}) \Gamma^k_{pm}(\mathbf{x}) \mathbf{g}^p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) + \tilde{v}_{p,m}(\mathbf{x}) \mathbf{g}^p(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) \\
&= \tilde{v}_{k|m}(\mathbf{x}) \mathbf{g}^k(\mathbf{x}) \otimes \mathbf{g}^m(\mathbf{x}) \qquad (28)
\end{aligned}$$

where the covariant derivative $\tilde{v}_{k|m}$ of the covariant components of $\mathbf{v}$ is given by (**??**) .
Divergence. The divergence of a vector field is now easily computed from the results (**??**) and (**??**). Specifically, in terms of the contravariant coordinates for $\tilde{\mathbf{v}}(\mathbf{x})$, one has from (**??**) and (**??**) that

$$\begin{aligned}
\mathrm{Div}(\tilde{\mathbf{v}}(\mathbf{x})) &= \mathrm{tr}\,(\nabla \tilde{\mathbf{v}}(\mathbf{x})) \\
&= (\tilde{v}^p_{,m}(\mathbf{x}) + \tilde{v}^k(\mathbf{x}) \Gamma^p_{km}(\mathbf{x})) \mathbf{g}_p(\mathbf{x}) \cdot \mathbf{g}^m(\mathbf{x}) \\
&= \tilde{v}^m_{,m}(\mathbf{x}) + \tilde{v}^k(\mathbf{x}) \Gamma^m_{km}(\mathbf{x}). \qquad (29)
\end{aligned}$$

On the other hand, in terms of the covariant coordinates for $\tilde{\mathbf{v}}(\mathbf{x})$, one obtains from (**??**) and (**??**) that

$$\mathrm{Div}(\tilde{\mathbf{v}}(\mathbf{x})) = g^{kn}(\mathbf{x}) \left[\tilde{v}_{k,n}(\mathbf{x}) - \tilde{v}_r(\mathbf{x}) \Gamma^r_{kn}(\mathbf{x}) \right]. \qquad (30)$$

It follows from (**??**), that

$$\begin{aligned}
\mathrm{Div}(\mathbf{g}^k(\mathbf{x})) = \mathrm{tr}(\nabla \mathbf{g}^k(\mathbf{x})) &= g^{km}(\mathbf{x}) \mathbf{g}_{m,n}(\mathbf{x}) \cdot \mathbf{g}^n(\mathbf{x}) + g^{km}_{,m}(\mathbf{x}) \\
&= g^{km}(\mathbf{x}) \Gamma^n_{mn}(\mathbf{x}) + g^{km}_{,m}(\mathbf{x}) \\
&= -g^{mn}(\mathbf{x}) \Gamma^k_{nm}(\mathbf{x}), \qquad (31)
\end{aligned}$$

while from (??), one sees that

$$\mathrm{Div}(\mathbf{g}_k(\mathbf{x})) = \mathrm{tr}(\nabla \mathbf{g}_k(\mathbf{x})) = \Gamma^m_{km}(\mathbf{x}). \tag{32}$$

It is also useful to derive formulas for the divergence of second order tensor fields, $\tilde{T}(\mathbf{x})$. For example, suppose $\tilde{T}(\mathbf{x})$ has the mixed contravariant-covariant representation $\tilde{T}(\mathbf{x}) = \tilde{T}^k_l(\mathbf{x})\mathbf{g}_k(\mathbf{x}) \otimes \mathbf{g}^l(\mathbf{x})$. Then, recalling the definition of $\mathrm{Div}(\tilde{T}(\mathbf{x}))$ as the unique vector satisfying $\mathrm{Div}(\tilde{T}(\mathbf{x})^T\mathbf{a}) = \mathrm{Div}(\tilde{T}(\mathbf{x})) \cdot \mathbf{a}$ for all constant vectors $\mathbf{a} \in \mathbb{R}^N$, one computes that

$$\mathrm{Div}(\tilde{T}(\mathbf{x})) \cdot \mathbf{a} = \mathrm{Div}(\tilde{T}(\mathbf{x})^T\mathbf{a}) = \mathrm{Div}(\tilde{v}_k(\mathbf{x})\mathbf{g}^k(\mathbf{x}))$$

with

$$\tilde{v}_k(\mathbf{x}) := \tilde{T}^l_k(\mathbf{x})\left(\mathbf{g}_l(\mathbf{x}) \cdot \mathbf{a}\right). \tag{33}$$

Making use of the result (??), one obtains

$$\begin{aligned}
\mathrm{Div}(\tilde{T}(\mathbf{x})^T\mathbf{a}) &= \tilde{T}^l_k(\mathbf{x})\left(g^{km}(\mathbf{x})\Gamma^p_{mp}(\mathbf{x}) + g^{kp}_{,p}(\mathbf{x})\right)(\mathbf{g}_l(\mathbf{x}) \cdot \mathbf{a}) + \\
&\quad g^{kp}(\mathbf{x})\left(\tilde{T}^l_{k,p}(\mathbf{x})(\mathbf{g}_l(\mathbf{x}) \cdot \mathbf{a}) + \tilde{T}^l_{k,p}(\mathbf{x})(\mathbf{g}_{l,p}(\mathbf{x}) \cdot \mathbf{a})\right) \\
&= \mathbf{a} \cdot \Big[\tilde{T}^l_k(\mathbf{x})\left(g^{km}(\mathbf{x})\Gamma^p_{mp}(\mathbf{x}) + g^{kp}_{,p}(\mathbf{x})\right)\mathbf{g}_l(\mathbf{x}) + \\
&\quad g^{kp}(\mathbf{x})\tilde{T}^l_{k,p}(\mathbf{x})\mathbf{g}_l(\mathbf{x}) + \tilde{T}^n_k(\mathbf{x})\Gamma^l_{np}(\mathbf{x})\mathbf{g}_l(\mathbf{x})\Big] \\
&= \mathbf{a} \cdot \mathrm{Div}(\tilde{T}(\mathbf{x}))
\end{aligned} \tag{34}$$

with

$$\begin{aligned}
\mathrm{Div}(\tilde{T}(\mathbf{x})) &= \left[\tilde{T}^l_k(\mathbf{x})\left(g^{km}(\mathbf{x})\Gamma^p_{mp}(\mathbf{x}) + g^{kp}_{,p}(\mathbf{x})\right) + g^{kp}(\mathbf{x})\left(\tilde{T}^l_{k,p}(\mathbf{x}) + \tilde{T}^n_k(\mathbf{x})\Gamma^l_{np}(\mathbf{x})\right)\right]\mathbf{g}_l(\mathbf{x}) \\
&= g^{lp}(\mathbf{x})\left[\tilde{T}^k_{l,p}(\mathbf{x}) + \tilde{T}^n_l(\mathbf{x})\Gamma^k_{np}(\mathbf{x}) - \tilde{T}^k_n(\mathbf{x})\Gamma^n_{lp}(\mathbf{x})\right]\mathbf{g}_k(\mathbf{x}).
\end{aligned} \tag{35}$$

Laplacian. The Laplacian, $\triangle$, is another commonly arising differential operator. For a scalar field, $\tilde{\phi}(\mathbf{x})$ one has

$$\begin{aligned}
\triangle\tilde{\phi}(\mathbf{x}) = \mathrm{Div}(\nabla\tilde{\phi}(\mathbf{x})) &= \mathrm{Div}\left(\mathbf{g}^j(\mathbf{x})\tilde{\phi}_{,j}(\mathbf{x})\right) \\
&= \tilde{\phi}_{,j}(\mathbf{x})\mathrm{Div}(\mathbf{g}^j(\mathbf{x})) + \mathbf{g}^j(\mathbf{x}) \cdot \nabla\tilde{\phi}_{,j}(\mathbf{x}) \\
&= \tilde{\phi}_{,j}(\mathbf{x})\left(g^{jm}(\mathbf{x})\Gamma^n_{mn}(\mathbf{x}) + g^{jm}_{,m}(\mathbf{x})\right) + \mathbf{g}^j(\mathbf{x}) \cdot (\mathbf{g}^k(\mathbf{x})\tilde{\phi}_{,jk}(\mathbf{x})) \\
&= \tilde{\phi}_{,j}(\mathbf{x})\left(g^{jm}(\mathbf{x})\Gamma^n_{mn}(\mathbf{x}) + g^{jm}_{,m}(\mathbf{x})\right) + g^{jk}(\mathbf{x})\tilde{\phi}_{,jk}(\mathbf{x}) \\
&= g^{jk}(\mathbf{x})\left[\tilde{\phi}_{,jk}(\mathbf{x}) - \tilde{\phi}_{,l}(\mathbf{x})\Gamma^l_{jk}(\mathbf{x})\right].
\end{aligned} \tag{36}$$

The Laplacian can also be defined for a vector field $\tilde{\mathbf{v}}(\mathbf{x})$ as

$$\triangle\tilde{\mathbf{v}}(\mathbf{x}) := \mathrm{Div}(\nabla\tilde{\mathbf{v}}(\mathbf{x})). \tag{37}$$

If the vector field is expressed through its contravariant representation $\tilde{\mathbf{v}}(\mathbf{x}) = \tilde{v}^k(\mathbf{x})\mathbf{g}_k(\mathbf{x})$, then the reader should verify that its Laplacian takes the form

$$\begin{aligned}
\triangle\tilde{\mathbf{v}}(\mathbf{x}) &= \mathbf{g}_j(\mathbf{x})g^{mn}(\mathbf{x})\left[\tilde{v}^j_{,mn}(\mathbf{x}) + 2\tilde{v}^r_{,n}(\mathbf{x})\Gamma^j_{rm}(\mathbf{x}) - \tilde{v}^j_{,r}(\mathbf{x})\Gamma^r_{mn}(\mathbf{x})\right. \\
&\quad \left.\tilde{v}^s(\mathbf{x})\left(\Gamma^r_{sm}(\mathbf{x})\Gamma^j_{rn}(\mathbf{x}) - \Gamma^j_{sr}(\mathbf{x})\Gamma^r_{mn}(\mathbf{x})\right) + \tilde{v}^r\Gamma^j_{rm,n}(\mathbf{x})\right].
\end{aligned} \tag{38}$$

On the other hand, if the vector field is expressed through its covariant representation $\tilde{\mathbf{v}}(\mathbf{x}) = \tilde{v}_k(\mathbf{x})\mathbf{g}^k(\mathbf{x})$, then the reader should verify that its Laplacian takes the form

$$\begin{aligned} \triangle\tilde{\mathbf{v}}(\mathbf{x}) &= \mathbf{g}_k(\mathbf{x}) \left[g^{pl}(\mathbf{x})g^{jk}(\mathbf{x})\tilde{v}_{j,lp}(\mathbf{x}) - \left(2g^{pl}(\mathbf{x})g^{jk}(\mathbf{x})\Gamma^n_{jl}(\mathbf{x}) + g^{jl}(\mathbf{x})g^{nk}(\mathbf{x})\Gamma^p_{lj}(\mathbf{x})\right)\tilde{v}_{n,p}(\mathbf{x}) \right. \\ & \left. + g^{pl}(\mathbf{x})g^{nk}(\mathbf{x})\left(\Gamma^j_{lp}(\mathbf{x})\Gamma^s_{nj}(\mathbf{x}) + \Gamma^j_{np}(\mathbf{x})\Gamma^j_{jl}(\mathbf{x}) - \Gamma^s_{nl,p}(\mathbf{x})\right)\tilde{v}_s(\mathbf{x})\right]. \end{aligned} \tag{39}$$

3.1 Cylindrical Coordinates

The general presentation given in the previous section is applied herein to the cylindrical coordinate system in $\mathbb{R}^3$. Adopting the notation in (**??**) and (**??**) one shows for a scalar field $\tilde{\phi}(r, \theta, z)$, that the gradient is given by

$$\nabla\tilde{\phi}(r, \theta, z) = \mathbf{j}_1(\theta)\partial_r\tilde{\phi} + \frac{1}{r}\mathbf{j}_2(\theta)\partial_\theta\tilde{\phi} + \mathbf{j}_3\partial_z\tilde{\phi}. \tag{40}$$

whereas the Laplacian takes the form

$$\triangle\tilde{\phi}(r, \theta, z) = \tilde{\phi}_{,rr} + \frac{1}{r}\tilde{\phi}_{,r} + \frac{1}{r^2}\tilde{\phi}_{,\theta\theta} + \tilde{\phi}_{,zz}. \tag{41}$$

Suppose that $\tilde{\mathbf{v}}(r, \theta, z)$ is a vector field with component representation $\tilde{\mathbf{v}}(r, \theta, z) = \tilde{u}^k(r, \theta, z)\mathbf{j}_k(\theta)$. Then its gradient has the form

$$\nabla\tilde{\mathbf{v}}(r, \theta, z) = g^{mn}(r, \theta, z)\mathbf{j}_m(\theta) \otimes \mathbf{j}_n(\theta) \tag{42}$$

with

$$\begin{array}{lll} g^{11} = \partial_r\tilde{u}^1 & g^{12} = \frac{1}{r}(\partial_\theta\tilde{u}^1 - \tilde{u}^2) & g^{13} = \partial_z\tilde{u}^1 \\ g^{21} = \partial_r\tilde{u}^2 & g^{22} = \frac{1}{r}(\partial_\theta\tilde{u}^2 + \tilde{u}^1) & g^{23} = \partial_z\tilde{u}^2 \\ g^{31} = \partial_r\tilde{u}^3 & g^{32} = \frac{1}{r}\partial_\theta\tilde{u}^3 & g^{33} = \partial_z\tilde{u}^3. \end{array} \tag{43}$$

From (**??**) and (**??**) it follows immediately that

$$\mathrm{Div}(\tilde{\mathbf{v}}(r, \theta, z)) = \mathrm{tr}(\nabla\tilde{\mathbf{v}}) = \partial_r\tilde{u}^1 + \frac{1}{r}\tilde{u}^1 + \frac{1}{r}\partial_\theta\tilde{u}^2 + \partial_z\tilde{u}^3. \tag{44}$$

3.2 Spherical Coordinates

The general results are next applied to the spherical coordinate system in $\mathbb{R}^3$. Adopting the notation in (**??**) and (**??**), one obtains for a scalar field $\tilde{f}(\rho, \theta, \phi)$ the result

$$\nabla\tilde{f}(\rho, \theta, \phi) = \tilde{f}_{,\rho}\mathbf{k}_1 + \frac{\tilde{f}_{,\theta}}{\rho\sin(\phi)}\mathbf{k}_2 + \frac{\tilde{f}_{,\phi}}{\rho}\mathbf{k}_3. \tag{45}$$

For a vector field $\tilde{\mathbf{v}}(\rho, \theta, \phi) = \tilde{v}^j\mathbf{k}_j(\theta, \phi)$, the reader should verify that

$$\mathrm{Div}(\tilde{\mathbf{v}}) = v^1_{,\rho} + \frac{2}{\rho}v^1 + \frac{1}{\rho\sin(\phi)}v^2_{,\theta} + \frac{1}{\rho}v^3_{,\phi} + \frac{\cos(\phi)}{\rho\sin(\phi)}v^3. \tag{46}$$

Finally, the reader should derive for the Laplacian applied to the scalar field $\tilde{f}(\rho, \theta, \phi)$ the result

$$\triangle\tilde{f} = \frac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho\tilde{f}) + \frac{1}{\rho^2\sin(\phi)^2}\partial_{\theta\theta}\tilde{f} + \frac{1}{\rho^2\sin(\phi)}\partial_\phi(\sin(\phi)\partial_\phi\tilde{f}). \tag{47}$$