Finding Steklov eigen solutions by boundary element method

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Consider finding Steklov eigen solutions (u, λ) such that

$$\begin{cases} (-\Delta + a^2 I)u(x) = 0 & \text{in } \Omega \\ \frac{\partial u(x)}{\partial \nu} = \lambda u(x) & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain with a smooth boundary $\partial\Omega$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator, I is the identity.

Let E be the fundamental solution defined by

$$(-\Delta_x + aI)E(|\xi - x|) = \delta(\xi - x)$$

where δ is the delta function. It is known

$$E(|\xi - x|) = \begin{cases} -\frac{1}{2\pi}\ln|\xi - x| & (n=2), \quad (4\pi|\xi - x|)^{-1} \quad (n=3), \quad a = 0, \\ \frac{1}{2\pi}K_0(\sqrt{a}|\xi - x|) & (n=2), \quad \frac{e^{-\sqrt{a}|\xi - x|}}{4\pi|\xi - x|} \quad (n=3), \quad a > 0, \end{cases}$$

where K_0 is the modified Bessel function of order 0. Then we have a simple-layer potential representation (SLPR) [2] for u satisfying $(-\Delta + aI)u = 0$, i.e.,

$$u(x) = \int_{\partial\Omega} E(|\xi - x|)\eta(\xi)\, d\sigma_\xi \equiv (L_b\eta)(x), \quad \forall x \in \mathbb{R}^n \tag{2}$$

in layer density η. For such u, by a known jump-discontinuity, we have (*weakly singular*)

$$\frac{\partial u}{\partial \nu}(x) = \frac{1}{2}\eta(x) + \int_{\partial\Omega} \frac{\partial E(|\xi - x|)}{\partial \nu_x}\eta(\xi)\, d\sigma_\xi \equiv (\partial_\nu L_b\eta)(x), \quad \forall x \in \partial\Omega. \tag{3}$$

In the above, L_b and $\partial_\nu L_b$ are two linear boundary integral operators. So the eigen solution problem leads to solve eigen solution (η, λ) such that

$$(\partial_\nu L_b\eta)(x) = \lambda(L_b\eta)(x) \quad x \in \partial\Omega. \tag{4}$$

We partition $\partial\Omega = \cup_{j=1}^n \partial\Omega_j$ where each $\partial\Omega_j$ is a line segment centered at x_j. Let $\eta(x) = \eta_j$ for $x \in \partial\Omega_j$, $\eta = (\eta_1, ..., \eta_n)$. Denote the matrices $A = (a_{ij})_{n\times n}$ and $DA = (da_{ij})_{n\times n}$ by

$$a_{ij} = \int_{\partial\Omega_j} E(|\xi - x_i|)d\sigma_\xi$$

$$da_{ij} = \frac{1}{2}\delta_{ij} + \int_{\partial\Omega_j} \frac{\partial}{\partial \nu_x} E(|\xi - x_i|) d\sigma_\xi,\ \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

We have

$$\begin{aligned} \frac{\partial}{\partial \nu_x} E(|\xi - x|) &= \tfrac{-1}{2\pi|\xi-x|^2}[(\xi - x)\cdot \nu_x] \qquad \text{for } E(|\xi - x|) = \frac{-1}{2\pi}\ln(|\xi - x|); \\ \frac{\partial}{\partial \nu_x} E(|\xi - x|) &= \tfrac{-\sqrt{a}}{2\pi} K_1(\sqrt{a}|\xi - x|)\tfrac{(\xi-x)\cdot\nu_x}{|\xi - x|} \quad \text{for } E(|\xi - x|) = \frac{1}{2\pi}K_0(\sqrt{a}|\xi - x|). \end{aligned}$$

where K_1 is the modified Bessel function of order 1. The integrals in a_{ij} and da_{ij} can be efficiently evaluated by, e.g., the Gaussian quadrature [2]. In particular, when $i = j$ and x_i, ξ are on the same line segment $\partial\Omega_i$, we have $(\xi - x_i)\cdot \nu_x = 0$. So there will not be any singularity in numerical evaluation of the terms da_{ij}.

Then eigen solution problem (4) becomes a linear $n \times n$-matrix eigen solution problem

$$DA\,\eta = \lambda A\,\eta. \tag{5}$$

There are many subroutines available to solve such a problem, e.g., we use Matlab subroutine $\mathbf{eig}(\mathbf{DA}, \mathbf{A})$ Once (η^i, λ^i) is found, then

$$u_b^i = A\eta^i$$

is the restriction of the eigen function u^i on $\partial\Omega$. While u^i on Ω can be evaluated by

$$u^i(x) = \sum_{j=1}^{n} \eta_j^i \int_{\partial\Omega_j} E(|\xi - x|) d\sigma_\xi \quad \forall x \in \Omega.$$

We evaluate u^i on an automatically generated finite element mesh so a solution profile plot can be easily generated by a Matlab subroutine **pdemesh**. This finite element mesh is used only for the plot. So it is not necessary to be refined.

Steps to use the Matlab Code:

(1) Open a Matlab window and go to the directory where the package files are put.

(2) In a Matlab command window, type "set_problem" and carry out the interactive input-outputs. Once it asks to export (g, b), go to the PDEToolBox window, click $\partial\Omega$, then click "Boundary", select "Export Decomposed Geometry, Boundary Conditions" and click "O.K." The problem setting and some preliminary works are done.

(3) Then just type "solve" in the command window and follow the interactive input-outputs.

References

[1] G. Auchmuty, Steklov eigenproblems and the representation of solutions of elliptic boundary value problems, *Numer. Func. Anal. Optimization*, 25(2004) 321-348.

[2] G. Chen and J. Zhou, *Boundary Element Methods*, Academic Press, London–San Diego, 1992.