Homework 3 Solutions

pages 279-283

1a) False. Counterexample: Any diagonal matrix with some diagonal entry appearing more than once.

d) True. If \( \lambda_1 + \lambda_2 \) are distinct, but \( E_{\lambda_1} \cap E_{\lambda_2} \neq \{0\} \), then \( \exists v \neq 0 \) so that \( v \in E_{\lambda_1} \cap E_{\lambda_2} \). Then \( \lambda_1 \neq \lambda_2 \) is a contradiction.

f) False. The backwards direction doesn't hold generally. If \( \chi_T \) does not split, you can have \( \text{mult}(\lambda) = \dim(E_\lambda) \neq \lambda \), but \( T \) is not diagonalizable.

\[ T(0) = 0, \quad T(x) = 1, \quad T(x^2) = 2x+2, \quad T(x^3) = 3x^2+6x, \]
so the matrix is \[
\begin{pmatrix}
0 & 1 & 2 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
so \( \chi_T = \det(A - \lambda I) = \lambda^4 \), which clearly splits.

But, \( \text{rank}(A - \lambda I) = 3 \) while \( \text{mult}(\lambda) = 4 \), so \( T \) isn't diagonalizable.

b) \( V = P_2(\mathbb{R}) \) and \( T(ax^2+bx+c) = cx^2+bx+a \). If \( \beta = \{1, x, x^2, x^3\} \), then \( T(1) = x^2, \quad T(x) = x, \quad T(x^2) = 1 \), so \( [T]_\beta = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \).

Then \( \chi_T = \det(A - \lambda I) = (1-\lambda)(\lambda^2-1) \), which splits.

The eigenvalues are \( \lambda_1 = 1, \lambda_2 = -1 \).

\( \lambda_1 = 1 \) has multiplicity 2, and \( \text{rank} \begin{pmatrix}
-1 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & -1
\end{pmatrix} = 1 = 3-2 \checkmark \)

\( \lambda_2 = -1 \) has multiplicity 1, and \( \text{rank} \begin{pmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 1
\end{pmatrix} = 2 = 3-1 \checkmark \)

so \( T \) is diagonalizable.

Let \( \gamma = \{x, 1+x^2, 1-x^2\} \). Then \( [T]_\gamma = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \).
7. Let \( A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2 \times 2}(\mathbb{R}) \).

\( \chi_A = (\lambda - 5)(\lambda + 1) \), so \( A \) has eigenvalues \( \lambda_1 = 5 \) and \( \lambda_2 = -1 \).
We can calculate that \( \{(1)\} \) is a basis for the eigenvalue \( \lambda_1 \) and \( \{(-2)\} \) is a basis for the eigenvalue \( \lambda_2 \).

Now let \( Q = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \). Then \( A = QDQ^{-1} \) where \( D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \).

\[ A^n = QD^nQ^{-1} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix} \]

\[ = \begin{pmatrix} 5^n & -2(-1)^n \\ 5^n & (-1)^n \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5^n + 2(-1)^n & 2 \cdot 5^n - 2(-1)^n \\ 5^n - (-1)^n & 2 \cdot 5^n + (-1)^n \end{pmatrix} \]

(See Example 7 page 272 for another problem like this one.)

9. a) Let \( T \) be a linear operator on a finite-dimensional vector space \( V \), and suppose \( \beta \) is an ordered basis for \( V \) such that \( [T]_\beta \) is upper triangular. Then \( ([T]_\beta - \lambda I) \) is also upper triangular, so its determinant is the product of its diagonal entries, say \( (a_{11} - \lambda) \cdots (a_{nn} - \lambda) \), where \( n = \dim(V) \). But we can use any basis \( \beta \) for \( V \), and get \( \chi_T = \det([T]_\beta - \lambda I) \), by the remark on page 249. So \( \chi_T \) splits.

b) Statement: Suppose \( A \) is similar to an upper triangular matrix, \( B \). Then \( \chi_A \) splits.

Proof: By above, \( \chi_B \) splits, since \( B \) is upper triangular.
But similar matrices have the same characteristic polynomial, and thus \( \chi_A \) splits also.
Let $T: V \rightarrow V$ be an invertible linear transformation. 

a) Recall if $\lambda$ is an eigenvalue for $T$, then $\lambda^{-1}$ is an eigenvalue for $T^{-1}$. Let $E_{\lambda} = \{ v \in V \mid T v = \lambda v \}$ and $E_{\lambda^{-1}} = \{ y \in V \mid T^{-1} y = \lambda^{-1} y \}$. We'll show these are equal:

If $v \in E_{\lambda}$, then $T v = \lambda v \Rightarrow T^{-1}(T v) = T^{-1}(\lambda v) \Rightarrow v = \lambda T^{-1}(v) \Rightarrow \lambda^{-1} v = T^{-1} v \Rightarrow v \in E_{\lambda^{-1}}$.

If $y \in E_{\lambda^{-1}}$, then $T^{-1} y = \lambda^{-1} y \Rightarrow T(T^{-1} y) = T(\lambda^{-1} y) \Rightarrow y = \lambda^{-1} (T y) \Rightarrow \lambda y = Ty \Rightarrow y \in E_{\lambda}$. So $E_{\lambda} = E_{\lambda^{-1}}$.

b) Suppose $T$ is diagonalizable. So for some basis $\beta$, $[T]_\beta$ is diagonal. So $T = Q [T]_\beta Q^{-1}$, for some invertible matrix $Q$. (These are both $n \times n$ matrices, where $n = \dim V$).

Then $T^{-1} = (Q [T]_\beta Q^{-1})^{-1} = Q^{-1} [T]^{-1}_\beta Q$. But if $[T]_\beta$ is diagonal, so is $[T]^{-1}_\beta$, so this proves $T^{-1}$ is diagonalizable.

Let $V$ be a f.d. v.s. and $W_1, \ldots, W_k$ subspaces so that $V = \sum W_i$.

Suppose $V = W_1 \oplus \cdots \oplus W_k$. By Thm 5.10 e, there ordered bases $Y_i$ of $W_i$ ($i = 1, \ldots, k$) so that $Y_1 \cup \cdots \cup Y_k$ is an ordered basis for $V$. Since $W_i \cap (\sum_{j \neq i} W_j) = \emptyset$, $i \in \{1, \ldots, k\}$, we then have

$\#(Y_1 \cup \cdots \cup Y_k) = \sum_{i=1}^k \#(Y_i)$, so $\dim V = \sum_{i=1}^k \dim(W_i)$.

Suppose $\dim V = \sum_{i=1}^k \dim(W_i)$. Let $Y_1, \ldots, Y_k$ be ordered bases for $W_1, \ldots, W_k$, respectively. Then $Y_1 \cup \cdots \cup Y_k$ span $V$, clearly. Suppose (WLOG) $\exists y_i \in Y_i$ that can be written in terms of the rest of the elements in $Y_1, \ldots, Y_k$. So $y_i = \sum c_{ij} y_j$. Then

$\dim(W_1 + \sum_{j=2}^k W_j) < \dim(W_1) + \dim(\sum_{j=2}^k W_j) \leq \dim(W_1) + \cdots + \dim(W_k)$.

But then $\dim V = \dim(W_1 + \cdots + W_k) < \dim(W_1) + \cdots + \dim(W_k)$, which contradicts our assumption. So $Y_1 \cup \cdots \cup Y_k$ is a basis for $V$, and $V = W_1 \oplus \cdots \oplus W_k$ by Thm 5.10 (d).