Homework 3 Solutions

pages 279-283

① a) False. Counterexample: Any diagonal matrix with some diagonal entry appearing more than once.

d) True. If λ_1 + λ_2 are distinct, but $E_{\lambda_1} \cap E_{\lambda_2} \neq \{0\}$, then $\exists$ $v \neq 0$ so that $v \in E_{\lambda_1} \cap E_{\lambda_2}$. Then $\lambda_1 v = \lambda_2 v$, + so $\lambda_1 = \lambda_2$, a contradiction.

f) False. The backwards direction doesn't hold generally. If χ_T does not split, you can have $\text{mult}(\lambda) = \dim(E_\lambda)$ $\forall$ λ, but T not diagonalizable.

③ a) Let $V = P_3(\mathbb{R})$ and $T(f(x)) = f'(x) + f''(x)$.

$T(0) = 0$, $T(x) = 1$, $T(x^2) = 2x + 2$, $T(x^3) = 3x^2 + 6x$,

so the matrix is $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A = [T]_\beta$. $(\beta = \{1, x, x^2, x^3\})$.

So $\chi_T = \det(A - \lambda I) = \lambda^4$, which clearly splits.

But, $\text{rank}(A - \lambda I) = 3$ while $\text{mult}(\lambda) = 4$, so T isn't diagonalizable.

b) $V = P_2(\mathbb{R})$ and $T(ax^2 + bx + c) = cx^2 + bx + a$. If $\beta = \{1, x, x^2\}$, then $T(1) = x^2$, $T(x) = x$, $T(x^2) = 1$, so $[T]_\beta = A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Then $\chi_T = \det(A - \lambda I) = (1 - \lambda)(\lambda^2 - 1)$, which splits.

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$.

$\lambda_1 = 1$ has multiplicity 2, and $\text{rank}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = 1 = 3 - 2$ ✓.

$\lambda_2 = -1$ has multiplicity 1, and $\text{rank}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2 = 3 - 1$. ✓

So T is diagonalizable.

Let $\gamma = \{x, 1 + x^2, 1 - x^2\}$. Then $[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. ∎

⑦ Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

$\chi_A = (\lambda-5)(\lambda+1)$, so A has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$.
We can calculate that $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ is a basis for the eigenvalue λ_1 and $\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$ is a basis for the eigenvalue λ_2.

Now let $Q = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$. Then $A = QDQ^{-1}$ where $D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\Rightarrow A^n = QD^nQ^{-1} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 5^n & -2(-1)^n \\ 5^n & (-1)^n \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5^n + 2(-1)^n & 2\cdot 5^n - 2(-1)^n \\ 5^n - (-1)^n & 2\cdot 5^n + (-1)^n \end{pmatrix} \blacksquare$$

(see Example 7 page 272 for another problem like this one.)

⑨ a) Let T be a linear operator on a f.d.v.s. V, + suppose ∃ ordered basis β such that $[T]_\beta$ is upper triangular. Then $([T]_\beta - \lambda I)$ is also upper triangular so its determinant is the product of its diagonal entries, say $(a_{11}-\lambda)\cdots(a_{nn}-\lambda)$, where $n = \dim(V)$. But we can use any basis β for V, and get $\chi_T = \det([T]_\beta - \lambda I)$, by the remark on page 249. So χ_T splits.

b) Statement: Suppose $A \in M_{n\times n}(F)$ is similar to an upper triangular matrix, B. Then χ_A splits.

Pf: By above, χ_B splits, since B is upper triangular. But similar matrices have the same characteristic polynomial, + thus χ_A splits also. $\blacksquare$

(12) Let $T: V \to V$ be an invertible linear transformation.

a) Recall if λ is an eigenvalue for T, then λ^{-1} is an eigenvalue for T^{-1}. Let $E_\lambda = \{v \in V \mid Tv = \lambda v\}$ and $E_{\lambda^{-1}} = \{y \in V \mid T^{-1}y = \lambda^{-1} y\}$. We'll show these are equal:

If $v \in E_\lambda$, then $Tv = \lambda v \Rightarrow T^{-1}(Tv) = T^{-1}(\lambda v)$

$\Rightarrow v = \lambda T^{-1}(v) \Rightarrow \lambda^{-1} v = T^{-1} v \Rightarrow v \in E_{\lambda^{-1}}$. ✓

If $y \in E_{\lambda^{-1}}$, then $T^{-1} y = \lambda^{-1} y \Rightarrow T(T^{-1} y) = T(\lambda^{-1} y)$

$\Rightarrow y = \lambda^{-1}(Ty) \Rightarrow \lambda y = Ty \Rightarrow y \in E_\lambda$. So $E_\lambda = E_{\lambda^{-1}}$.

b) Suppose T is diagonalizable. So for some basis β, $[T]_\beta$ is diagonal. So $T = Q([T]_\beta)Q^{-1}$, for some invertible matrix Q. (These are both $n \times n$ matrices, where $n = \dim V$).

Then $T^{-1} = (Q[T]_\beta Q^{-1})^{-1} = Q^{-1}[T]_\beta^{-1} Q$. But if $[T]_\beta$ is diagonal, so is $[T]_\beta^{-1}$, so this proves T^{-1} is diagonalizable. ■

(20) Let V be a f.d.v.s. and $W_1, \dots, W_k$ subspaces so that $V = \sum_1^k W_i$.

$\boxed{\Rightarrow}$ Suppose $V = W_1 \oplus \cdots \oplus W_k$. By Thm 5.10e, $\exists$ ordered bases γ_i of W_i $(i = 1, \dots, k)$ so that $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for V. Since $W_i \cap (\sum_{j \neq i} W_j) = \{0\}$ $\forall i \in \{1, \dots, k\}$, we then have $\#(\gamma_1 \cup \cdots \cup \gamma_k) = \sum_{i=1}^k \#(\gamma_i)$, so $\dim V = \sum_{i=1}^k \dim(W_i)$

$\boxed{\Leftarrow}$. Suppose $\dim V = \sum_{i=1}^k \dim(W_i)$. Let $\gamma_1, \dots, \gamma_k$ be ordered bases for $W_1, \dots, W_k$, respectively. Then $\gamma_1 \cup \cdots \cup \gamma_k$ span V, clearly. Suppose (WLOG) $\exists\, \gamma_{11} \in \gamma_1$ that can be written in terms of the rest of the elements in $\gamma_1, \dots, \gamma_k$. So $\gamma_{11} = \sum_{\substack{i,j=1 \\ (i,j) \neq (1,1)}}^k c_{ij} \gamma_{ij}$. Then

$\dim(W_1 + \sum_{j=2}^k W_j) < \dim(W_1) + \dim(\sum_2^k W_j) \leq \dim(W_1) + \cdots + \dim(W_k)$.

But then $\dim V = \dim(W_1 + \cdots + W_k) < \dim(W_1) + \cdots + \dim(W_k)$, which contradicts our assumption. So $\gamma_1 \cup \cdots \cup \gamma_k$ is a basis for V, + $V = W_1 \oplus \cdots \oplus W_k$ by Thm 5.10(d) ■