Lemma 1 *Let V be an n dimensional vector space and $T \in Hom(V,V)$ of nilpotency index q. For all $1 \leq l \leq q$, there exists $\{\alpha_{l,i} : 1 \leq i \leq k_l\}$ such that, if $\Delta_{l,i} = \{\alpha_{l,i}, T\alpha_{l,i}, \ldots, T^{q-l}\alpha_{l,i}\}$, then $\cup_{1\leq r\leq l-1} \cup_{1\leq i\leq k_r} \Delta_{r,i}$ are linearly independent and*

$$\mathcal{R}(T^{q-l}) \subseteq sp(\cup_{1\leq r\leq l-1} \cup_{1\leq i\leq k_r} \Delta_{r,i}) + \mathcal{R}(T^{q-l}) \cap \mathcal{N}(T).$$

Proof. We'll prove this by induction. For $l = 1$ it is trivial, since $\mathcal{R}(T^{q-1}) \subseteq \mathcal{N}(T)$. Assume the result is true for some $l = 1, \ldots, q-1$. Extend the linearly independent set $\cup_{1\leq r\leq l-1} \cup_{1\leq i\leq k_r} \Delta_{r,i}$ to a basis of $\text{sp}(\cup_{1\leq r\leq l-1} \cup_{1\leq i\leq k_r} \Delta_{r,i}) + \mathcal{R}(T^{q-l}) \cap \mathcal{N}(T)$ with the new elements of this basis, $\{\beta_{l,i}\}_{i=1}^{k_l}$, in $\mathcal{R}(T^{q-l}) \cap \mathcal{N}(T)$. In particular, there exist $\{\alpha_{l,i}\}_{i=1}^{k_l}$ with $T^{q-l}\alpha_{l,i} = \beta_{l,i}$. It is easy to check that $\cup_{1\leq r\leq l} \cup_{1\leq i\leq k_r} \Delta_{r,i}$ are linearly independent and we have $\mathcal{R}(T^{q-l}) \subseteq \text{sp}(\cup_{1\leq r\leq l} \cup_{1\leq i\leq k_r} \Delta_{r,i})$.

Now take $x \in \mathcal{R}(T^{q-(l+1)})$. So, there exists y with $T^{q-(l+1)}y = x$. Then $Tx \in \mathcal{R}(T^{q-l})$ and hence there exists constants, $c_{r,i,j}$ with

$$T^{q-l}y = Tx = \sum_{r=1}^{l} \sum_{i=1}^{k_r} \sum_{j=0}^{q-r} c_{r,i,j} T^j \alpha_{l,i}.$$

Therefore $0 = \sum_{r=1}^{l} \sum_{i=1}^{k_r} \sum_{j=0}^{q-r} c_{r,i,j} T^{j+l} \alpha_{r,i}$. But if $j+l \geq q$ the term $T^{j+l}\alpha_{l,i}$ is zero. So,

$$T^{q-l}y = Tx = \sum_{r=1}^{l} \sum_{i=1}^{k_r} \sum_{j=0}^{q-l-1} c_{r,i,j} T^j \alpha_{l,i}.$$

Hence,

$$0 = \sum_{r=1}^{l} \sum_{i=1}^{k_r} \sum_{j=0}^{q-(l+1)} c_{r,i,j} T^{j+l} \alpha_{r,i}.$$

Therefore by linear independence, $c_{r,i,j} = 0$ for $1 \leq r \leq l, 1 \leq i \leq k_r, 0 \leq j \leq q-(l+1)$. We then get:

$$Tx = T^{q-l}y = T\big(T^{q-(l+1)}\big(\sum_{r=1}^{l} \sum_{i=1}^{k_r} \sum_{j=q-l}^{q-r} c_{r,i,j} T^{j-(q-l)} \alpha_{l,i}\big)\big).$$

Therefore,

$$x - \big(\textstyle\sum_{r=1}^{l} \sum_{i=1}^{k_r} \sum_{j=q-l}^{q-r} c_{r,i,j} T^{j-1} \alpha_{l,i}\big)$$
$$= T^{q-(l+1)}y - T^{q-(l+1)}\big(\textstyle\sum_{r=1}^{l} \sum_{i=1}^{k_r} \sum_{j=q-l}^{q-r} c_{r,i,j} T^{j-(q-l)} \alpha_{l,i}\big) \in \mathcal{N}(T) \cap \mathcal{R}(T^{q-(l+1)}).$$

Corollary 1 *Let V be an n dimensional vector space and $T \in Hom(V,V)$ of nilpotency index q. For all $1 \leq l \leq q$, there exists $\{\alpha_{l,i} : 1 \leq i \leq k_l\}$ such that, if $\Delta_{l,i} = \{\alpha_{l,i}, T\alpha_{l,i}, \ldots, T^{q-l}\alpha_{l,i}\}$, then $\cup_{1\leq r\leq q} \cup_{1\leq i\leq k_r} \Delta_{r,i}$ are linearly independent and*

$$V = sp(\cup_{1\leq r\leq q} \cup_{1\leq i\leq k_r} \Delta_{r,i}).$$