Theorem 1 *Here is a proof of the Jordan decomposition theorem for nilpotent linear operators with index* 3.

Proof. We first notice that for all $x \in V$, $T(T^2x) = 0$. Hence, $\mathcal{R}(T^2) \subseteq \mathcal{N}(T)$. So, $\mathcal{R}(T^2) \cap \mathcal{N}(T) = \mathcal{R}(T^2)$.

Now, take any point $x \in \mathcal{R}(T)$. Then, $Tx \in \mathcal{R}(T^2)$. Let $\beta_1 = \{b_{1,1}, b_{1,2}, \ldots, b_{1,k_1}\}$ be a basis for $\mathcal{R}(T^2)$. Since each $b_{1,j} \in \mathcal{R}(T^2)$, there exists $a_{1,j} \in V$ with $T^2 a_{1,i} = b_{1,i}$. Let $\Delta_{1,i} := \{a_{1,i}, Ta_{1,i}, T^2 a_{1,i}\}$, the last element of this set being $b_{1,j}$ and β_1 is a basis for $\mathcal{R}(T^2)$. We had already shown that $\bigcup_{i=1}^{k_1} \Delta_{1,i}$ is a linearly independent set. So, this last set (which contains β_1) is a basis for its span, which contains $\mathcal{R}(T^2)$. Now,

$$Tx = \sum_{i=1}^{k_1} c_{1,i} T^2 a_{1,i} = T\big(\sum_{i=1}^{k_1} c_{1,i} T a_{1,i}\big).$$

Hence, $T\big(x - \sum_{i=1}^{k_1} c_{1,i} T a_{1,i}\big) = 0$. So, $x - \sum_{i=1}^{k_1} c_{1,i} T a_{1,i} \in \mathcal{N}(T)$, and both x and $\sum_{i=1}^{k_1} c_{1,i} T a_{1,i}$ are in $\mathcal{R}(T)$. Therefore,

$$x - \sum_{i=1}^{k_1} c_{1,i} T a_{1,i} \in \mathcal{R}(T) \cap \mathcal{N}(T).$$

Since $\sum_{i=1}^{k_1} c_{1,i} T a_{1,i} \in \text{sp}(\bigcup_{i=1}^{k_1})$, we have

$$x \in \text{sp}(\bigcup_{i=1}^{k_1} \Delta_{1,i}) + \Big(\mathcal{R}(T) \cap \mathcal{N}(T)\Big).$$

Since this worked for every $x \in \mathcal{R}(T)$, we have

$$\mathcal{R}(T) \subseteq \text{sp}(\bigcup_{i=1}^{k_1} \Delta_{1,i}) + \Big(\mathcal{R}(T) \cap \mathcal{N}(T)\Big).$$

Now choose a basis for the right-hand side which extends $\text{sp}(\bigcup_{i=1}^{k_1} \Delta_{1,i})$, for which the new elements, call them $\{b_{2,1}, \ldots, b_{2,k_2}\}$, are in $\Big(\mathcal{R}(T) \cap \mathcal{N}(T)\Big)$. Since each $b_{2,i}$ is in $\Big(\mathcal{R}(T) \cap \mathcal{N}(T)\Big) \subseteq \mathcal{R}(T)$, for each $i \leq k_2$ there is an $a_{2,i}$ with $Ta_{2,i} = b_{2,i}$. As before, we let $\Delta_{2,i} := \{a_{2,i}, Ta_{2,i}\}$. Then, from the Lemma we proved the other day, we know that $\bigcup_{l=1}^{2} \bigcup_{i=1}^{k_l} \Delta_{l,i}$ is a linearly independent set, which is a basis for $\text{sp}(\bigcup_{i=1}^{k_1} \Delta_{1,i}) + \Big(\mathcal{R}(T) \cap \mathcal{N}(T)\Big)$. So,

$$\mathcal{R}(T) \subseteq \bigcup_{l=1}^{2} \bigcup_{i=1}^{k_l} \Delta_{l,i} = \bigcup_{l=1}^{2} \bigcup_{i=1}^{k_l} \{a_{l,i}, Ta_{l,i}, \ldots, T^{3-l} a_{l,i}\}.$$

Now, take any $z \in V$ and consider Tz which is automatically in $\mathcal{R}(T)$. As before write $Tz = \sum_{l=1}^{2} \sum_{i=1}^{k_l} \sum_{j=0}^{3-l} c_{l,i,j} T^j a_{l,i}$. So, as before, applying T^2 (last time it was T) we get

$$0 = T^3 z = \sum_{l=1}^{2} \sum_{i=1}^{k_l} \sum_{j=0}^{3-l} c_{l,i,j} T^{j+2} a_{l,i}.$$

The terms with $j + 2 \geq 3$ are zero. So, the above equals

$$\sum_{l=1}^{2} \sum_{i=1}^{k_l} c_{l,i,0} T^2 a_{l,i}.$$

If $l = 2, T^2 a_{2,i} = T b_{2,i} = 0$. So, we can get rid of those terms also to get

$$\sum_{i=1}^{k_1} c_{1,i,0} T^2 a_{1,i} = 0.$$

By the linear independence of $\{T^2 a_{1,i} : i = 1, \ldots, k_1\}, c_{1,i,0} = 0$. Plugging that into the formula above for Tz we get

$$Tz = \sum_{i=1}^{k_2} \sum_{j=0}^{3-2} c_{2,i,j} T^j a_{2,i}.$$