Math 423 **Quiz 3** **Name:** Solutions

J. Zinn **ID#** ________________

1. Let $\beta = \{p_0, p_1, p_2\}$ be the usual basis for $\mathcal{P}_2$. Define the linear transformation $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ by
$$T(p) = p + p_1 p' + p_2 p''.$$
Find the matrix of T with respect to β.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$
   (a) Find the eigenvalues of A.
   (b) Find the eigenspaces corresponding to the eigenvalues.
   (c) Find a basis of $\mathcal{R}^3$ consisting of eigenvectors.
   (d) Write the matrix of A with respect to the basis of eigenvectors you just found.

3. Prove that if A is nilpotent (with nilpotency index $= q$), then the only eigenvalue for A is $\lambda = 0$.

① $T(p_0) = T(1) = 1 + x \cdot 0 + x^2 \cdot 0 = 1 = p_0$
$T(p_1) = T(x) = x + x \cdot 1 + x^2 \cdot 0 = 2x = 2p_1$
$T(p_2) = T(x^2) = x^2 + x \cdot 2x + x^2 \cdot 2 = 5x^2 = 5p_2$
So $[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$

② This problem had a typo making the computations a little tougher than planned. For a similar problem worked out, see Example 6 page 270, or page 2 of these solutions.

③ Suppose $A^q = 0$ but $A^{q-1} \neq 0$. Suppose λ is an eigenvalue for A. Then by Homework 2 (p.259 #15), λ^q is an eigenvalue for A^q. Then, $\exists$ nonzero vector v so that $A^q v = \lambda^q v$. But $A^q = 0$, so then $0 = \lambda^q v$. Then, since $v \neq 0$, $\lambda^q = 0 \Rightarrow \lambda = 0$, as desired. ∎

(See next page for Quiz 3 Addendum) →

Math 423 **Quiz 3(addendum)** **Name:** Solutions

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1. Let $A = \begin{bmatrix} 3 & -1 & -1 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}$
   (a) Find the eigenvalues of A.
   (b) Find the eigenspaces corresponding to the eigenvalues.
   (c) Find a basis of $\mathcal{R}^3$ consisting of eigenvectors.
   (d) Write the matrix of A with respect to the basis of eigenvectors you just found.

a)
$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & -1 & -1 \\ 0 & 4-\lambda & 1 \\ 0 & 2 & 5-\lambda \end{pmatrix} = (3-\lambda)\left[(4-\lambda)(5-\lambda) - 2\right] + 0 + 0.$$
$$= (3-\lambda)(18 - 9\lambda + \lambda^2) = (3-\lambda)(\lambda-3)(\lambda-6)$$

So the eigenvalues of A are $\lambda_1 = 3$ (multiplicity 2)
$\lambda_2 = 6$ (multiplicity 1).

b) $E_{\lambda_1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \right\}$

which is the solution to the system $x_2 + x_3 = 0$, + so we can write $E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$.

$E_{\lambda_2} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -3 & -1 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \right\}$, which is the solution to the system $\begin{cases} 3x_1 + x_2 + x_3 = 0 \\ -2x_2 + x_3 = 0 \end{cases}$, so we can write

$E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$.

c) We've seen in b) that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ are all eigenvectors. It is straightforward to check that these three vectors are a basis for $\mathbb{R}^3$.

d) Let $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$. Since $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$, and $A\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ -12 \end{pmatrix}$, we have $[A]_B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.