**Math 423** **Quiz 1** **Name:**__________________________

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**Definition 1** *The null space of a linear operator $T : V_1 \longrightarrow V_2$ is*

$$\{x \in V : T(x) = 0\}.$$

**Theorem 2** *$T : C^n[0,1] \longrightarrow C[0,1]$ given by $T(f) = f^{(n)}$ is a linear transformation.*

**Proof.** Assume that $f, g \in C[0,1]$ and $\alpha, \beta$ are scalars.
To show that: $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$.
In other words we have to show that $(\alpha f + \beta g)^{(n)} = \alpha f^{(n)} + \beta g^{(n)}$. But, the derivative of a constant times a function is the constant times the derivative of the function. This can be iterated to get the same result for any order of the derivative (of course, you could officially use induction).

**Theorem 3** *If $T : V_1 \longrightarrow V_2$ is linear and $M$ is a subspace (but not necessarily all) of the null space $N \subseteq V_1$, then*

$$\tilde{T} : V_1/M \longrightarrow V_2$$

*given by $\tilde{T}(M + x) = T(x)$ is well-defined.*

**Proof.** Assume that $M+x = M+y$ and show that $T(x) = \tilde{T}(M+x) = \tilde{T}(M+y) = T(y)$. But, we've proved in class that $M + x = M + y$ is equivalent to $x - y \in M$. Since, $M \subseteq N$, we know that $x - y \in N$. Since, $N$ is the null space (or kernel) of $T$, we get that $T(x - y) = 0$. Since $T$ is linear, $T(x) - T(y) = T(x - y) = 0$. Hence, $T(x) = T(y)$. This is what we had to show.

**Theorem 4** *(i) If $t_0 \in [0,1]$ and $S : C[0,1] \longrightarrow \mathbb{R}$ is given by $S(f) = f(t_0)$, then $S$ is linear.*

*(ii) If $T : C[0,1] \longrightarrow C[0,1]$ is given by $T(f)(u) = (1-u)f(0) + uf(1)$ is a linear transformation.*

*(iii) Using the operator defined in 4(ii) find the nullspace of $T$ and the range (this is more difficult) of $T$.*

**Proof of (i).** $S(\alpha f + \beta g) = (\alpha f + \beta g)(t_0) =$ ( by definition $\alpha f(t_0) + \beta g(t_0) = \alpha S(f) + \beta S(g)$. Hence, $S$ is a linear transformation.
**Proof of (ii).** We have to show that $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$. Translating this to our specific $T$, we have to show that for every $u \in [0,1]$,

$$(1-u)(\alpha f + \beta g)(0) + u(\alpha f + \beta g)(1) = \alpha((1-u)f(0) + uf(1)) + \beta((1-u)g(0) + ug(1)).$$

The left-hand side is equal to $(1-u)(\alpha f(0) + \beta g(0)) + u(\alpha f(1) + \beta g(1))$. And this equals (by using commutativity and associativity of real numbers) $\alpha((1-u)f(0) + uf(1)) + \beta((1-u)g(0) + ug(1))$, which is what we needed to prove.
**Proof of (iii).** The null space of $T = \{f \in C[0,1] : T(f) = 0\}$. But, $T(f) = 0$ (as a function in $C[0,1]$) means that $T(f)(u) = 0$ for every $u \in [0,1]$. Or, that $(1-u)f(0) + uf(1) = 0$ for every $u \in [0,1]$. But, in particular, it would have to be 0 for $u = 0$ and for $u = 1$. In these two cases we have $f(0) = 0$ and $f(1) = 0$. So, the null space is a subset of $\{f \in C[0,1] : f(0) = f(1) = 0\}$. On the other hand, if $f(0) = f(1) = 0$, then $(1-u)f(0) + uf(1) = 0$ for every $u \in [0,1]$, so that $T(f) = 0$. Hence, the null space is $\{f \in C[0,1] : f(0) = f(1) = 0\}$.
Now, for the part about the range.

Since $f(0)$ and $f(1)$ can be any numbers, the range is $\{h \in C[0,1] : \exists a, b$ such that $h(u) = (1-u)a + ub, \forall u \in [0,1]\}$. Simplifying $(1-u)a + ub = a + (b-a)u$, again, since $a$ and $b - a$ are arbitrary, this is just the equation of a straight line. So, the range is just the straight lines (restricted to the interval $[0,1]$).