First one has to show that

$$d(\sum_{j=1}^{k} U_j) \leq \sum_{j=1}^{k} d(U_j).$$

Think of this as a homework assignment.

Proof. [2.19] We'll do this by induction. The m^{th} statement is: S_m: The two statements

$$\begin{aligned} V =& U_1 + U_2 + U_3 + \cdots + U_m \text{ and} \\ d(V) =& d(U_1) + d(U_2) + d(U_3) + \cdots + d(U_m) \end{aligned}$$

imply

$$V = U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_m.$$

We already proved this (implication) for $m = 2$ (and we also did $m = 3$).

We'll show that if we assume that S_{m-1} is true, then S_m is also true.

Let $W = U_1 + U_2 + U_3 + \cdots + U_{m-1}$. Then, using Theorem 2.18 and the assumption (that is,

$$\begin{aligned} V =& U_1 + U_2 + U_3 + \cdots + U_m \text{ and} \\ d(V) =& d(U_1) + d(U_2) + d(U_3) + \cdots + d(U_m) \end{aligned}$$

$$\sum_{j=1}^{m} d(U_j) = d(W + U_m) = d(W) + d(U_m) - d(W \cap U_m).$$

Cancelling $d(U_m)$ we get:

$$\begin{aligned} \sum_{j=1}^{m-1} d(U_j) &= d(W) - d(W \cap U_m) \leq d(W) \\ &= d(\sum_{j=1}^{m-1} U_j) \leq \text{ by your hw assignment } \leq \sum_{j=1}^{m-1} d(U_j). \end{aligned}$$

Since the left side equals the right side, all the things inequalities must be equalities. Now we recall that when we have equality in Theorem 2.18, the minus term is zero and the quantity inside the minus term equals (0). So, we have two things:

- $d(W) = d(\sum_{j=1}^{m-1} U_j) = \sum_{j=1}^{m-1} d(U_j)$ and
- $W \cap U_m = (0)$.

By induction we can conclude from the first of these conditions that

$$W = U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_{m-1}.$$

Now we'll show that if $x_1 + \cdots + x_{m-1} + x_m = 0$, where $x_j \in U_j$, then all the x_j's are zero. To do this we just use the "move it to the other side" trick. Namely,

$$x_1 + \cdots + x_{m-1} = -x_m.$$

So, both sides belong to $W \cap U_m$, which equals (0). Hence,

$$x_1 + \cdots + x_{m-1} = 0 \text{ also } x_m = 0.$$

But, since

$$W = U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_{m-1},$$

all the x_j's are zero for $j = 1, \ldots, m-1$. Also, $x_m = 0$. So, all the x_j's are zero. Therefore,

$$U_1 + U_2 + U_3 + \cdots + U_m = U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_m.$$

So, we've completed the induction. Therefore all the statements S_m are true.