Solutions - Exam 1 423 7/22/08

3. (a) $(1,0,0) \in$ the set, $(0,1,0) \in$ " $\Big\}$ But, $(1,0,0)+(0,1,0)=(1,1,0) \notin$ the set. $\therefore$ not a subsp.

(b)
$$\begin{bmatrix} a & b & a+b \\ a & a+c & c \\ b+c & b & c \end{bmatrix} + \begin{bmatrix} u & v & u+v \\ u & u+w & w \\ v+w & v & w \end{bmatrix} = \begin{bmatrix} a+u & b+v & (a+u)+(b+v) \\ a+u & (a+u)+(c+w) & c+w \\ (b+v)+(c+w) & b+v & c+w \end{bmatrix}$$
$\in$ the set.

Also, constant multip. works.
$\therefore$ a subsp.

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4. Fix $w \in V$. Show $\exists x \in V$ s.t. $T^3(x)=w$.
First, since T is onto, $\exists y \in V$, $T(y)=w$.
And, now $\exists z \in V$, $T(z)=y$. And, now $\exists x \in V$, $T(x)=z$. $\therefore$ $T^3(x)=T^2(Tx)=T^2(z)=T(Tz)$
$=T(y)=w$. $\therefore$ Every $w \in V$
is "hit". $\therefore$ T^3 is onto

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5. (a) $\ni$ $T(x)=0$ since $\{b_j\}$ is a bases of V, $\exists \{t_j\}$,
$\underline{x=t_1b_1+\cdots+t_nb_n}$. $\therefore$ $0=T(x)=t_1T(b_1)+\cdots+t_nT(b_n)$
(by linearity of T) $=(t_1 2)b_1+(t_2 2^2)b_2+\cdots+(t_n 2^n)b_n$
Since $\{b_j\}$ are lin. indep.
$2t_1=0$, $2^2t_2=0$, $\cdots$, $2^nt_n=0$. $\therefore t_1=0, \cdots, t_n=0$.
$\therefore$ $\underline{x=0}$.

(b) Since V is f.dim. & T is 1-1, T is also onto.

6. We know that if $V = U + W$, then

$d(V) = \dim(U) + d(W) - d(U \cap W) \leq \dim(U) + d(W)$ since $d(U \cap W) \geq 0$.

$\therefore$ if we have $V = (U + W) + Z$

$d(U) \leq d(U + W) + d(Z) \leq [d(U) + d(W)] + d(Z)$ $\boxtimes$

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7. $\hat{T}(x) = M + x$, $\ker(T) = \{x \in V : \hat{T}(x) = 0\}$

$= \{x \in V : M + x = M + 0\}$ (since $M+0$ is the zero of V/M)

By the $y - x$ Lemma, $M + x = M + 0$ iff $x = x - 0 \in M$. $\therefore \ker(T) = \{x \in V : x \in M\} = M$. $\boxtimes$

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8. ~~If~~ SHOW $T^3 = I$, then T is an isomorphism.

$T^3 = T(T^2) = I$ $\therefore$ T^2 is a right inverse of T. (If V is fin. dim, then we're done, because of our Theorem which says ~~on~~ a f. dim. vect. sp. a rt inv is a left inv).

But, for any V, $T^3 = T^2(T) = I$

$\therefore$ T^2 is a left inverse as well.

$\therefore$ $T^{-1} = T^2$.