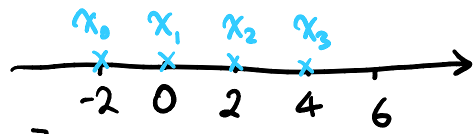
Week in Review 12

Monday, November 21, 2022 5:01 PM

Q1. $\Delta x = \frac{b-a}{n} = \frac{6-(-2)}{4} = \frac{8}{4} = 2.$



$$A \approx L_4 = \left[f(x_0) + f(x_1) + f(x_2) + f(x_3)\right]\Delta x$$

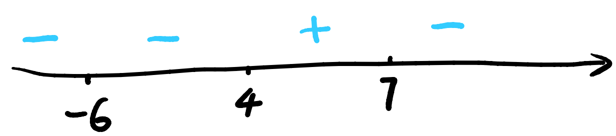
$$= \left[f(-2) + f(0) + f(2) + f(4)\right] \cdot 2$$

$$= \left[-12 - 4 + 4 + 60\right] \cdot 2$$

$$= \boxed{96}$$

Q2. $f'(x) = \frac{(x-4)^3 (x+6)^8}{7-x}$

check signs of f':



So f is increasing on $(4, 7)$.

Q3. $\lim\limits_{x \to \infty} x \sin\left(\frac{5}{x}\right)$ "$\infty \cdot 0$"

$$= \lim_{x \to \infty} \frac{\sin\left(\frac{5}{x}\right)}{\frac{1}{x}} \qquad \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x \to \infty} \frac{\cos\left(\frac{5}{x}\right) \cdot \left(-\frac{5}{x^2}\right)}{-\frac{1}{x^2}} \qquad \text{(L'Hopital's rule)}$$

$$= \lim_{x \to \infty} \cos\left(\frac{5}{x}\right) \cdot 5 = \cos(0) \cdot 5 = \boxed{5}$$

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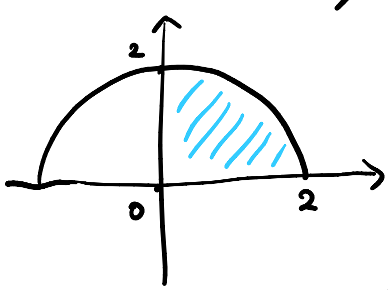
Q4. use geometry : $\int_0^2 \left(\sqrt{4-x^2} + 3x\right) dx$.

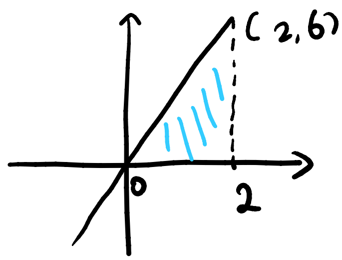
set $y = \sqrt{4-x^2} \geq 0 \Rightarrow y^2 = 4 - x^2 \quad (y \geq 0)$

$\Rightarrow x^2 + y^2 = 4 = 2^2. \quad (y \geq 0)$.



That's a quarter of a circle: $\frac{1}{4} \cdot \pi \cdot 2^2 = \pi$.

Next : set $y = 3x$. $0 \leq x \leq 2$.



It's a right triangle : $\frac{1}{2} \cdot 2 \cdot 6 = 6$

So $\int_0^2 \left(\sqrt{4-x^2} + 3x\right) dx = \boxed{\pi + 6}$.

Q5. $f(x) = \sqrt{2x} + 3$ on $[\frac{1}{2}, 2]$.

f is continuous on $[\frac{1}{2}, 2]$ and is differentiable on $(\frac{1}{2}, 2)$.

Compute $f'(x) = \sqrt{2} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{x}}$

and $\dfrac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = \dfrac{5-4}{\frac{3}{2}} = \dfrac{2}{3}$

Set $f'(c) = \frac{2}{3} \Rightarrow \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{c}} = \frac{2}{3}$

$3\sqrt{2}$ ... 9

$\Rightarrow \quad 2\sqrt{c} = 3$

$$\Rightarrow \quad \frac{3\sqrt{2}}{4} = \sqrt{c} \quad \Rightarrow \quad c = \frac{9}{8}$$

$c = \frac{9}{8}$ is in $(\frac{1}{2}, 2)$.

Q6. $$f'(x) = \frac{1}{x^2+1} - 5 - \sin(x).$$

So $$f(x) = \arctan(x) - 5x + \cos(x) + C$$

Use $f(1) = \cos(1)$: $$f(1) = \arctan(1) - 5\cdot 1 + \cos(1) + C$$
$$= \frac{\pi}{4} - 5 + \cos(1) + C$$

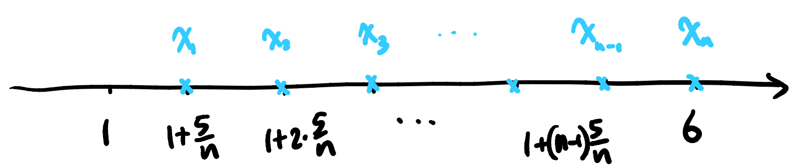
So $$\frac{\pi}{4} - 5 + \cos(1) + C = \cos(1).$$

Solve for $C = 5 - \frac{\pi}{4}$.

Therefore, $$f(x) = \arctan(x) - 5x + \cos(x) + 5 - \frac{\pi}{4}$$

Q7. $$\text{area} = \int_1^6 (x^3+5)^2\,dx = \lim_{n\to\infty} R_n$$

$$\Delta x = \frac{b-a}{n} = \frac{6-1}{n} = \frac{5}{n}$$



$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

note: $x_i = a + \Delta x \cdot i$
$= 1 + \frac{5}{n} i$

$$= \sum_{i=1}^{n} f\left(1 + \frac{5i}{n}\right) \frac{5}{n}$$

$$= \sum_{i=1}^{n} f\left(1+\frac{5i}{n}\right)\frac{5}{n}$$

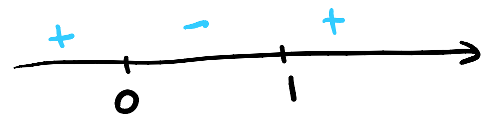
$$= \sum_{i=1}^{n} \left[\left(1+\frac{5i}{n}\right)^3+5\right]^2 \cdot \frac{5}{n}$$

$$\text{area} = \int_1^6 (x^3+5)^2\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1+\frac{5i}{n}\right)^3+5\right]^2 \cdot \frac{5}{n}$$

[remark: $L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} f(a+(i-1)\Delta x)\,\Delta x$]

Q8. $f''(x) = x^3\underbrace{(x^2+16)}_{>0}(x-1)$

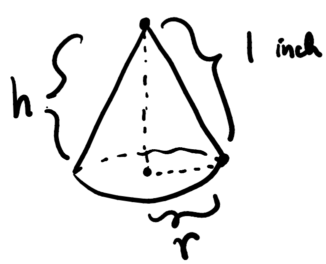
signs of f'':



f changes concavity at $x=0$ and $x=1$.

So $x=0$, $x=1$ are x-coordinates of inflection points.

Q9 Draw the diagram:



height : h

radius : r

maximize the volume : $V = \frac{1}{3}\pi r^2 h$

restriction : $h^2+r^2=1$

sub : $r^2 = 1-h^2$

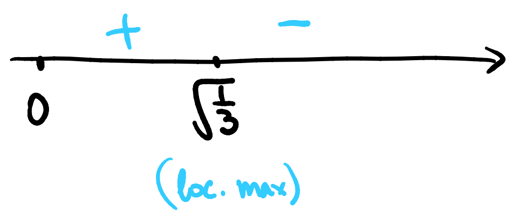
$$\Rightarrow \quad V(h) = \frac{1}{3}\pi(1-h^2)\cdot h$$

$$= \frac{\pi}{3}(h-h^3) \qquad (0<h<\infty)$$

Compute $V'(h) = \frac{\pi}{3}(1 - 3h^2)$

Set $V'(h) = 0 \Rightarrow \frac{\pi}{3}(1-3h^2)=0 \Rightarrow 1-3h^2=0$

Solve for h: $3h^2=1 \Rightarrow h^2=\frac{1}{3} \Rightarrow h=\sqrt{\frac{1}{3}}$.
$(h>0)$.



check the sign of V'

So $V(h)$ has a local max at $h=\sqrt{\frac{1}{3}}$.

So since it's the only local max. on $(0,\infty)$.

then it's the absolute max.

Dimensions: $h=\sqrt{\frac{1}{3}}$ and $r=\sqrt{\frac{2}{3}}$.

Q10.

(a) $\lim_{x\to\infty} \left[\ln(2x^2+9) - 3\ln(x+1) \right]$

$$= \lim_{x\to\infty} \left[\ln(2x^2+9) - \ln(x+1)^3 \right]$$

$$= \lim_{x\to\infty} \ln \frac{2x^2+9}{(x+1)^3}$$

$$= \ln\left(\lim_{x\to\infty} \frac{2x^2+9}{(x+1)^3} \right).$$

$\boxed{\begin{array}{l} a\ln b = \ln(b^a). \\ \ln A - \ln B = \ln \frac{A}{B} \end{array}}$

$\leftarrow$ " $\lim_{x\to\infty} \frac{2x^2}{x^3}$ "
$=$ " $\lim_{x\to\infty} 0^+$ "

$$= \ln(0^+)$$

$$= \boxed{-\infty}$$

(b) $\lim_{x\to 0^+} (1+\sin(x))^{\frac{3}{x^2}}$

$\boxed{\ln(a^b) = b\ln a}$

$= \lim_{x\to 0^+} e^{\ln(1+\sin(x))^{\frac{3}{x^2}}}$

$= \lim_{x\to 0^+} e^{\frac{3}{x^2}\ln(1+\sin(x))}$

$= e^{\lim_{x\to 0^+} \frac{3}{x^2}\ln(1+\sin(x))}$.

$= e^{\lim_{x\to 0^+} \frac{3\ln(1+\sin(x))}{x^2}}$ "$\frac{0}{0}$"

$= e^{\lim_{x\to 0^+} \frac{3\cdot\frac{1}{1+\sin(x)}\cdot\cos(x)}{2x}}$ "$\frac{3\cdot 1}{1+0} \big/ 2\cdot 0$" $=$ "$\frac{3}{0^+}$"

$= e^{\infty} = \boxed{\infty}$

(c) $\lim_{x\to 0^+} (2^x - 1)\csc(x)$ "$0\cdot\infty$"

$\boxed{\csc(x) = \frac{1}{\sin(x)}}$

$= \lim_{x\to 0^+} \frac{2^x - 1}{\sin(x)}$ "$\frac{0}{0}$"

$= \lim_{x\to 0^+} \frac{2^x\cdot\ln 2}{\cos(x)}$

$= \frac{2^0\cdot\ln 2}{\cos(0)} = \boxed{\ln 2}$

Q 11.

Q11.

(a) $f'(x) = 4^x + \sqrt[3]{x^4} - \frac{1}{7x} - \sin(x)$.

$\boxed{(4^x)' = 4^x \ln 4}$

$$f(x) = \frac{4^x}{\ln 4} + \frac{3}{7}x^{\frac{7}{3}} - \frac{1}{7}\ln|x| + \cos(x) + C$$

(b) $f'(x) = \frac{2x^3 - 7}{x^4} = \frac{2x^3}{x^4} - \frac{7}{x^4} = \frac{2}{x} - \frac{7}{x^4}$

$$f(x) = 2\cdot\ln|x| - 7\cdot\frac{x^{-3}}{(-3)} + C$$

$$= 2\ln|x| + \frac{7}{3}x^{-3} + C$$

(c) $f'(x) = x^2(8x-3) = 8x^3 - 3x^2$

$$f(x) = 2x^4 - x^3 + C$$

use $f(1) = -1$: $f(1) = 2\cdot 1^4 - 1^3 + C$

$= 1 + C$

$\Rightarrow \quad 1 + C = -1 \quad \Rightarrow \quad C = -2$

$$\boxed{f(x) = 2x^4 - x^3 - 2}$$

(d) $f'(x) = 4x^3 + \frac{6}{1+x^2} - 7$

$$f(x) = x^4 + 6\arctan(x) - 7x + C$$

use $f(0)=9$: $f(0) = C$

use $f(0)=9$: $f(0) = C$

$$\Rightarrow \quad C = 9$$

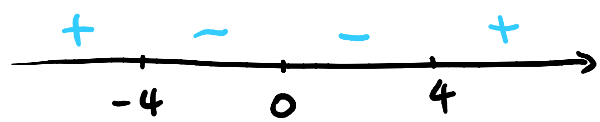
$$\boxed{f(x) = x^4 + 6\arctan(x) - 7x + 9}$$

Q 12. $f(x) = \frac{4}{x} + \frac{x}{4} + 2.$

(a) $x \neq 0$ so domain is $(-\infty, 0) \cup (0, \infty)$

(b) Compute $f'(x) = -\frac{4}{x^2} + \frac{1}{4} = \frac{-16 + x^2}{4x^2}$

check the sign of f':



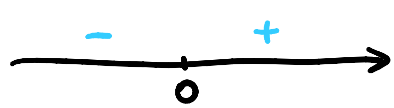
f is increasing on $(-\infty, -4)$ and on $(4, \infty)$

f is decreasing on $(-4, 0)$ and on $(0, 4)$.

f has local max at $x = -4$
f has local min at $x = 4$.

(c) Compute $f''(x) = \frac{8}{x^3}$

check sign of f'':



So f is concave up on $(0, \infty)$

f is concave down on $(-\infty, 0)$

f is concave down on $(-\infty, 0)$

(d) Look at $f(x) = \frac{4}{x} + \frac{x}{4} + 2$ on $[1,5]$

from (b) $f'(x) = \frac{-16+x^2}{4x^2}$

set up $f'(x) = 0 \Rightarrow x^2 - 16 = 0 \Rightarrow x = 4.$

x in $(1,5)$.

($x=-4$ and $x=0$ are ruled out).

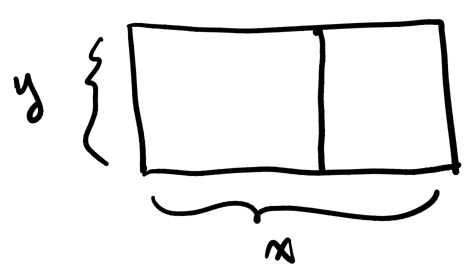
Compare $f(4) = \frac{4}{4} + \frac{4}{4} + 2 = 4$

$f(1) = \frac{4}{1} + \frac{1}{4} + 2 = \frac{25}{4}$

$f(5) = \frac{4}{5} + \frac{5}{4} + 2 = \frac{81}{20}$

f has global max $\frac{25}{4}$ and global min 4 on $[1,5]$.

Q13. Draw a diagram:



x: length y: width.

maximize: $A = xy$

restriction: $2x + 3y = 4800$

$\Rightarrow y = \frac{4800-2x}{3} = 1600 - \frac{2}{3}x$

$A(x) = x \cdot \left(1600 - \frac{2}{3}x\right)$

$= 1600x - \frac{2}{3}x^2 \quad (0 < x < \infty)$

Compute $A'(x) = 1600 - \frac{4}{3}x$

Set $A'(x)=0 \Rightarrow \frac{4}{3}x = 1600 \Rightarrow x = 1200$

Compute $A''(x) = -\frac{4}{3} < 0$. So $A''(1200) < 0$.

At $x = 1200$, $A(x)$ has a local max.

Since it is the only one local max.

$\Rightarrow$ $A(x)$ has a global max at $x = 1200$.

Dimensions: $x = 1200$ and $y = 800$.