Week in Review 1

Friday, August 19, 2022 10:19 AM

\# 1. Find the vector $\overrightarrow{AB}$.

(a) $A(1,3)$ and $B(4,4)$.

$$\overrightarrow{AB} = \langle 4-1,\ 4-3 \rangle$$

$$= \langle 3,\ 1 \rangle$$

$A(a_1, a_2)$ $B(b_1, b_2)$

$$\overrightarrow{AB} = \langle b_1 - a_1,\ b_2 - a_2 \rangle$$

(b) $A(3,-1)$ and $B(3,-3)$

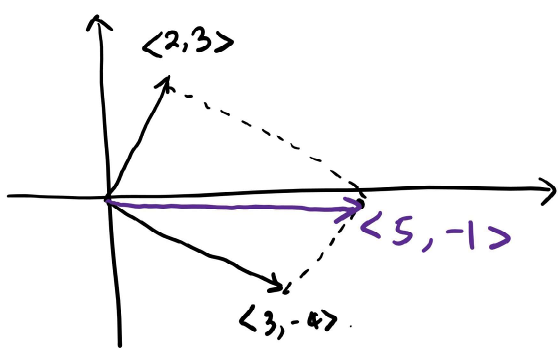
$$\overrightarrow{AB} = \langle 3-3,\ -3-(-1) \rangle$$

$$= \langle 0,\ -2 \rangle$$

\#2 Find sum of $\langle 2,3 \rangle$ and $\langle 3,-4 \rangle$. Sketch it.

Solution. $\langle 2,3 \rangle + \langle 3,-4 \rangle$

$$= \langle 5,\ -1 \rangle.$$



3. Given $\vec{a} = 5i - 12j$ and $\vec{b} = -2i + 8j$.

(a) Find $\|\vec{a}\|$.

$\vec{a} = \langle a_1, a_2 \rangle$.

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$.

$$\|\vec{a}\| = \sqrt{5^2 + (-12)^2}$$

$$= \sqrt{25 + 144} = \sqrt{169} = 13$$

(b) Find a unit vector in direction of $\vec{a}$.

Given $\vec{a} = \langle a_1, a_2 \rangle$

$\vec{u} = \dfrac{\vec{a}}{\|\vec{a}\|}$

By (a), we know $\|\vec{a}\| = 13$.

So unit vector denoted by

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{5i - 12j}{13} = \frac{5}{13}i - \frac{12}{13}j$$

$$\left(\text{or } \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle\right)$$

(c) Find $3\vec{a} + 4\vec{b}$.

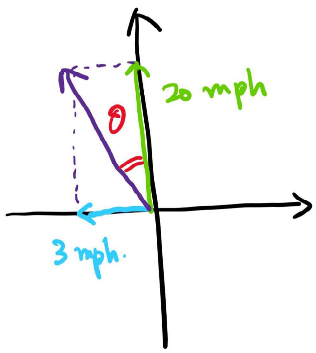
$\vec{a} = 5i - 12j$ and $\vec{b} = -2i + 8j$.

So $3\vec{a} + 4\vec{b} = 3(5i - 12j) + 4(-2i + 8j)$

$$= 15i - 36j + (-8i) + 32j$$

$$= 7i - 4j \qquad (\text{or } \langle 7, -4 \rangle).$$

#4.



To compute the speed :
we need to find the length of the resulting vector.
magnitude.

So $\text{speed} = \sqrt{3^2 + 20^2} = \sqrt{9+400}$

$= \sqrt{409}$ mph

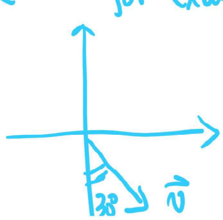
To represent the direction, we use the angle θ between the resulting vector and the direction in north/south

So $\tan\theta = \dfrac{3}{20} \Rightarrow \theta = \arctan\left(\frac{3}{20}\right)$.

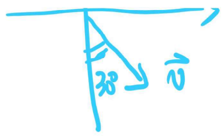
$\theta \approx 8.53^\circ$

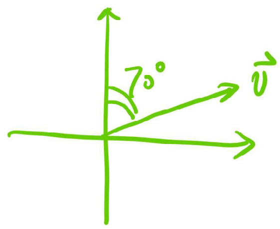
So the direction is : $N\ 8.53^\circ\ W$

remark: for example vector is pointing to south east



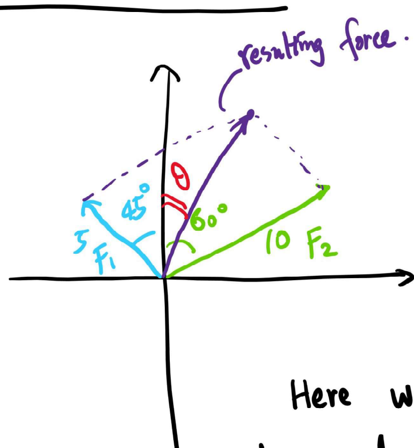
$S\ 30^\circ\ E$.



northeast : 

N 70° E

#5.



i horizontal , j vertical.

$$F_1 = -(\sin 45°) \cdot 5\, i + (\cos 45°)\, 5\, j$$

$$F_2 = (\sin 60°) \cdot 10\, i + (\cos 60°) \cdot 10\, j$$

Here we decompose F_1 and F_2 into horizontal direction (i) and vertical direction (j) respectively. (use trig. relation of right triangle)

So $$F_1 = -\frac{\sqrt{2}}{2} \cdot 5\, i + \frac{\sqrt{2}}{2} \cdot 5\, j$$

and $$F_2 = \frac{\sqrt{3}}{2} \cdot 10\, i + \frac{1}{2} \cdot 10\, j = 5\sqrt{3}\, i + 5\, j$$

So $$F = F_1 + F_2 = \underbrace{\left(-\frac{5\sqrt{2}}{2} + 5\sqrt{3}\right)}_{>0} i + \underbrace{\left(\frac{5\sqrt{2}}{2} + 5\right)}_{>0} j$$

magnitude of F : $$\|\vec{F}\| = \sqrt{\left(-\frac{5\sqrt{2}}{2} + 5\sqrt{3}\right)^2 + \left(\frac{5\sqrt{2}}{2} + 5\right)^2}$$

$$\approx 9.9558 \quad (lb).$$

direction of F : $$\tan\theta = \frac{-\frac{5\sqrt{2}}{2} + 5\sqrt{3}}{\frac{5\sqrt{2}}{2} + 5} \Rightarrow \theta = \arctan\left(\frac{-\frac{5\sqrt{2}}{2} + 5\sqrt{3}}{\frac{5\sqrt{2}}{2} + 5}\right)$$

direction of F : $\tan\theta = \dfrac{-\frac{5\sqrt{2}}{2}+5\sqrt{3}}{\frac{5\sqrt{2}}{2}+5} \Rightarrow \theta = \arctan\left(\dfrac{-\frac{5\sqrt{2}}{2}+5\sqrt{3}}{\frac{5\sqrt{2}}{2}+5}\right)$

$\approx 31^\circ$.

So the direction N31°E.

below it comes from J.2.

#6. Find $\vec{a}\cdot\vec{b}$

(a) $\vec{a} = \langle 2,3\rangle$ and $\vec{b} = \langle 3,-4\rangle$.

$\vec{a} = \langle a_1, a_2\rangle$
$\vec{b} = \langle b_1, b_2\rangle$

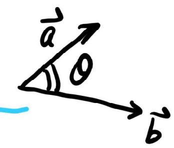
$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$

$\vec{a}\cdot\vec{b} = \langle 2,3\rangle \cdot \langle 3,-4\rangle$

$= 2\cdot 3 + 3\cdot(-4)$

$= 6 - 12 = -6$

(b) $\|\vec{a}\| = 3$, $\|\vec{b}\| = 4$, and the angle $\theta = \pi/3$.



$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

$= 3\cdot 4\cdot\cos\frac{\pi}{3}$

$= 3\cdot 4\cdot\frac{1}{2} = 6$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos\theta$

another formula to compute dot product.

#7. Find angle between $i + 3j$ and $2i - 4j$.

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2$$
$$= \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$
$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$$
$$\theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right).$$

Use the application :

$$\theta = \arccos\left(\frac{(i+3j)\cdot(2i-4j)}{\|i+3j\|\ \|2i-4j\|}\right)$$

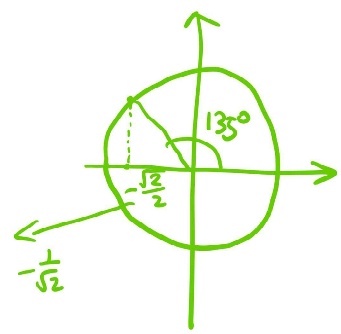
$$= \arccos\left(\frac{1\cdot 2 + 3\cdot(-4)}{\sqrt{1^2+3^2}\cdot\sqrt{2^2+(-4)^2}}\right)$$

$$= \arccos\left(\frac{-10}{\sqrt{10}\cdot\sqrt{20}}\right) = \arccos\left(\frac{-10}{\sqrt{200}}\right)$$

$$= \arccos\left(\frac{-10}{10\sqrt{2}}\right)$$

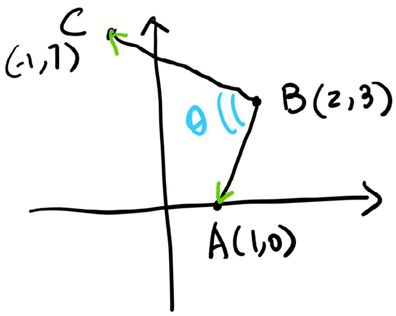
$$= \arccos\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{3\pi}{4} \quad (\text{or } 135^\circ).$$



Final answer $\theta = \frac{3\pi}{4}$.

#8. $A(1,0)$. $B(2,3)$. $C(-1,7)$. find angle $\angle ABC$.



Then we see angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is $\theta = \angle ABC$.

From coordinates, we see $\overrightarrow{BA} = \langle 1-2, 0-3\rangle = \langle -1,-3\rangle$

$$\vec{BC} = \langle -1-2, 7-3 \rangle = \langle -3, 4 \rangle$$

So $\theta = \arccos\left(\dfrac{\vec{BC} \cdot \vec{BA}}{\|\vec{BC}\| \, \|\vec{BA}\|}\right)$

$$= \arccos\left(\frac{\langle -3,4 \rangle \cdot \langle -1,-3 \rangle}{\|\langle -3,4 \rangle\| \, \|\langle -1,-3 \rangle\|}\right)$$

$$= \arccos\left(\frac{(-3)\cdot(-1) + 4\cdot(-3)}{\sqrt{(-3)^2+4^2} \cdot \sqrt{(-1)^2+(-3)^2}}\right)$$

$$= \arccos\left(\frac{-9}{5 \cdot \sqrt{10}}\right)$$

So the angle $\angle ABC = \arccos\left(\dfrac{-9}{5\sqrt{10}}\right)$.

\# 9. Determine if $\vec{a}$ and $\vec{b}$ are orthogonal:

$\vec{a}$ and $\vec{b}$ are orthogonal if the angle in between is $\frac{\pi}{2}$ (90°). $\Longleftrightarrow$ $\vec{a} \cdot \vec{b} = 0$

(a) $\vec{a} = \langle 3,1 \rangle$ and $\vec{b} = \langle -3,9 \rangle$.

verify: $\vec{a} \cdot \vec{b} = \langle 3,1 \rangle \cdot \langle -3,9 \rangle$

$$= 3\cdot(-3) + 1\cdot 9$$

$$= -9 + 9 = 0. \quad ✓$$

So $\vec{a}$ is orthogonal to $\vec{b}$.

(b) $\vec{a} = 2i - 7j$ and $\vec{b} = 5i + 3j$

verify: $\vec{a} \cdot \vec{b} = (2i - 7j) \cdot (5i + 3j)$

$$= 2 \cdot 5 + (-7) \cdot 3$$

$$= 10 - 21 = -11 \neq 0 .$$

So $\vec{a}$ and $\vec{b}$ are not orthogonal.

Below are the problems that haven't covered in the session

\# 10. Force $\vec{F} = i + 3j$ is used to move an object from $(2,3)$ to $(4,8)$. Find the work done by $\vec{F}$.

Solution. $\vec{s} = \langle 4-2, 8-3 \rangle = \langle 2, 5 \rangle$ displacement.

$$W = \vec{F} \cdot \vec{s} = \langle 1, 3 \rangle \cdot \langle 2, 5 \rangle$$

$$= 1 \cdot 2 + 3 \cdot 5 = 17 \quad (N \cdot m).$$

\# 11. Find scalar and vector projection of $\langle 3,1 \rangle$ onto $\langle 2,5 \rangle$.

Solution. $\text{comp}_{\langle 2,5 \rangle} \langle 3,1 \rangle = \dfrac{\langle 3,1 \rangle \cdot \langle 2,5 \rangle}{\| \langle 2,5 \rangle \|}$

Solution. $\text{comp}_{\langle 2,5\rangle}\langle 3,1\rangle = \dfrac{\langle 3,1\rangle \cdot \langle 2,5\rangle}{\|\langle 2,5\rangle\|}$

$$= \frac{3\cdot 2 + 1\cdot 5}{\sqrt{2^2+5^2}} = \frac{11}{\sqrt{29}}$$

$$\text{proj}_{\langle 2,5\rangle}\langle 3,1\rangle = \left(\frac{\langle 3,1\rangle \cdot \langle 2,5\rangle}{\|\langle 2,5\rangle\|^2}\right)\langle 2,5\rangle$$

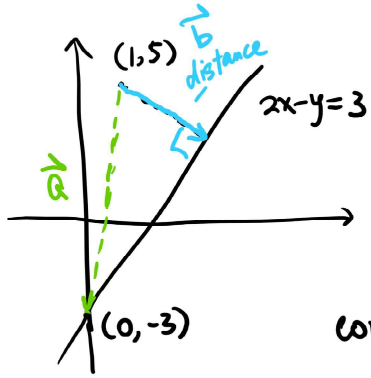
$$= \left(\frac{3\cdot 2 + 1\cdot 5}{2^2+5^2}\right)\langle 2,5\rangle$$

$$= \frac{11}{29}\langle 2,5\rangle$$

$$= \left\langle \frac{22}{29}, \frac{55}{29}\right\rangle$$

\# 12. Find distance from $(1,5)$ to line $2x-y=3$.

Solution.



Choose a point on the line $2x-y=3$: $(0,-3)$.

connect from $(1,5)$ to $(0,-3)$

$\vec{a} = \langle 0-1, -3-5\rangle = \langle -1, -8\rangle$

Find the slope of $2x-y=3$: $m=2$.

pick a direction parallel to it : $\langle 1, 2 \rangle$.

So the perpendicular direction is $\vec{b} = \langle -2, 1 \rangle$.

Use scalar projection to compute the distance:

$$\text{distance} = \left| \text{comp}_{\vec{b}} \vec{a} \right| = \left| \frac{\langle -1, -8 \rangle \cdot \langle -2, 1 \rangle}{\| \langle -2, 1 \rangle \|} \right|$$

$$= \left| \frac{2-8}{\sqrt{(-2)^2+1^2}} \right|$$

$$= \frac{6}{\sqrt{5}}$$