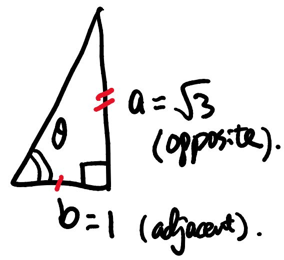
Week in Review 2

Monday, September 12, 2022 5:05 PM

1. Find the values.

(a) $\arctan(\sqrt{3})$.

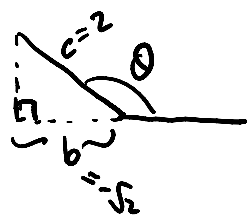


$$\tan\theta = \frac{a}{b}$$

$$\tan\theta = \sqrt{3} = \frac{\sqrt{3}}{1}$$

So $\arctan(\sqrt{3}) = \theta = 60^\circ$ or $\frac{\pi}{3}$.

(b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$



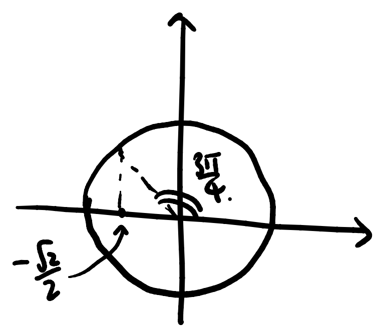
(negative cosine).

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}.$$

$$\cos\theta = \frac{-\sqrt{2}}{2}$$

So $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 135^\circ = \frac{3\pi}{4}$.

Alternatively, unit circle model:



(c) $\sin\left(2\sin^{-1}\left(\frac{3}{4}\right)\right)$.

Here we apply a formula for trig function:

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

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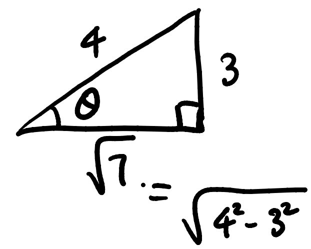
Let $\boxed{\theta = \sin^{-1}\left(\frac{3}{4}\right)}$, we see $\sin\theta = \frac{3}{4} = \frac{\text{opposite}}{\text{hypotenuse}}$.

Our goal is compute $\sin(2\theta) = 2\sin\theta\cos\theta$.

All we need is to find $\cos\theta$:



$\cos\theta = \frac{\sqrt{7}}{4}$.

So $\sin\left(2\sin^{-1}\left(\frac{3}{4}\right)\right) = \sin(2\theta)$

$= 2\sin\theta\cos\theta = 2\cdot\frac{3}{4}\cdot\frac{\sqrt{7}}{4} = \boxed{\frac{3\sqrt{7}}{8}}$

\# 2 Simplify the expressions.

(a) $\tan(\sin^{-1}(x))$.

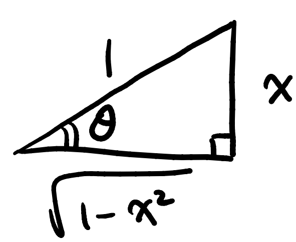
Let $\theta = \sin^{-1}(x)$. Our goal is find $\tan\theta$

So $\sin\theta = x$ so the question becomes:

we know $\sin\theta = x$, what is $\tan\theta = ?$

Now we use right triangle to assist us :

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}$



adjacent $\sqrt{1-x^2}$ $\sqrt{1-x^2}$

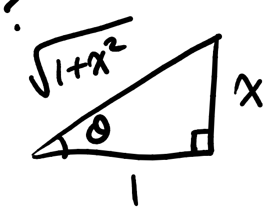
So $\tan(\sin^{-1}(x)) = \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$.

(b). $\cos(\arctan(x))$.

Let $\theta = \arctan(x)$. then $\tan\theta = x$

Then we need to find $\cos\theta = ?$

Now use right triangle to help :

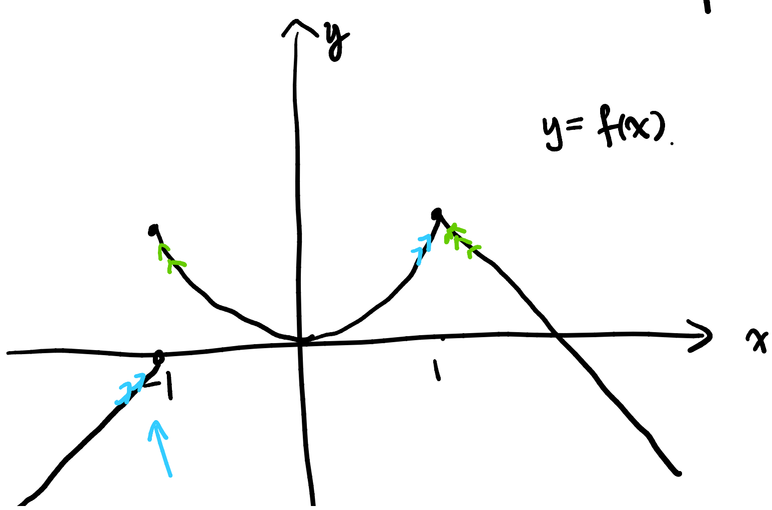


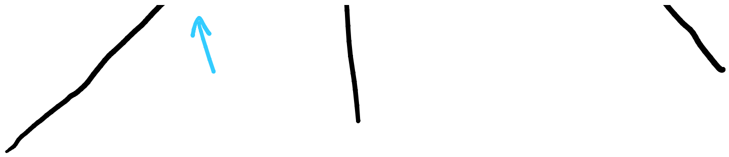
$$\cos\theta = \frac{\text{adjacent}}{\text{hypertunes}} = \frac{1}{\sqrt{1+x^2}}$$

So $\cos(\arctan(x)) = \cos\theta = \dfrac{1}{\sqrt{1+x^2}}$.

Q3. Sketch the graph of

$$f(x) = \begin{cases} 1+x & \text{if } x<-1 \\ x^2 & \text{if } -1\le x<1 \\ 2-x & \text{if } x\ge 1 \end{cases}$$





Find the limits :

$$\lim_{x\to -1^-} f(x) = 1+(-1) = 0.$$

$$\lim_{x\to -1^+} f(x) = (-1)^2 = 1$$

$$\lim_{x\to 1^-} f(x) = 1^2 = 1$$

$$\lim_{x\to 1^+} f(x) = 2-1 = 1$$

<1> Does $\lim_{x\to -1} f(x)$ exist? No , $\lim_{x\to -1} f(x) =$ DNE.

(because $\lim_{x\to -1^-} f(x) = 0 \neq 1 = \lim_{x\to -1^+} f(x)$).

<2> Does $\lim_{x\to 1} f(x)$ exist? Yes, $\lim_{x\to 1} f(x) = 1$

(because $\lim_{x\to 1^-} f(x) = 1 = \lim_{x\to 1^+} f(x)$).

#4. Find the limits.

(a) $$\lim_{x\to -3^-} \frac{x+1}{x+3}.$$

We observe that $(-3)+3=0$. So we cannot simply plug in $x=-3$.

numerator : $x+1 \to (-3)+1 = -2$ (as $x\to -3^-$)

Look at numerator: $x+1 \to (-3)+1=-2$. (as $x \to -3^-$)

Look at denominator: $x+3 \to (-3)+3=0$ (as $x \to -3^-$).

Since $x \to -3^-$ means $x<-3$ (from the left).

$$\Rightarrow \quad x+3<0.$$

$$\frac{x+1}{x+3} \to \frac{-2}{\substack{\text{negative} \\ (\text{small} \to 0)}} = \infty$$

So $\displaystyle\lim_{x \to -3^-} \frac{x+1}{x+3} = \infty$.

(b) $\displaystyle\lim_{x \to 5} \frac{x+1}{x-5}$.

Observation:

$x+1 \to 5+1=6$. (good).

$x-5 \to 5-5=0$ (not good for denominator).

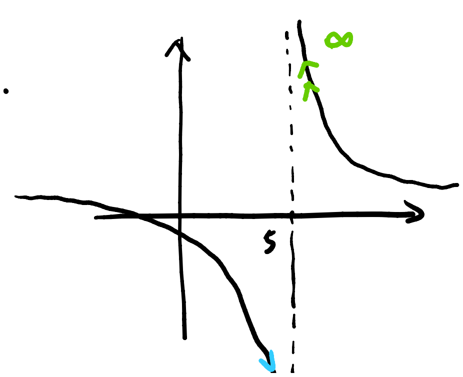
We need to look at 2 different sides:

$$\lim_{x \to 5^-} \frac{x+1}{x-5} = \frac{5+1}{\left(\substack{\text{negative} \\ \text{small} \to 0}\right)} = -\infty \quad \text{(left side).}$$

$$\lim_{x \to 5^+} \frac{x+1}{x-5} = \frac{5+1}{\left(\substack{\text{positive} \\ \text{small} \to 0}\right)} = \infty \quad \text{(right side).}$$

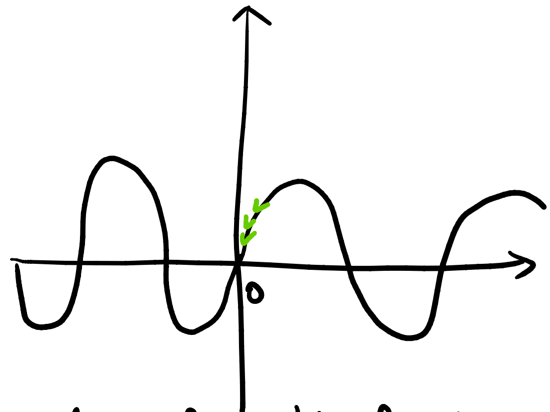
So they don't match (not the same).

Then $\displaystyle\lim_{x \to 5} \frac{x+1}{x-5} = DNE$.



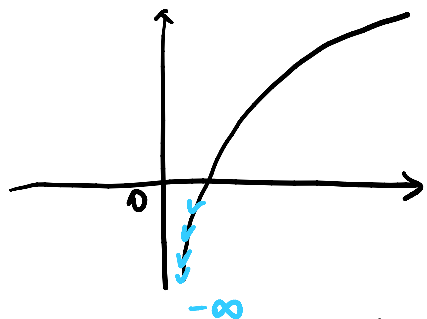
(c) $\lim_{x \to 0^+} \ln(\sin(x))$.

graph of $\sin(x)$:



$\lim_{x \to 0^+} \sin(x) = 0$. (approach to the right of 0).

graph of ln :



If the variable is approaching to 0 from the right, then $\ln(\cdot)$ is tending to $-\infty$.

So : $\lim_{x \to 0^+} \ln(\sin(x)) = -\infty$

(d) $\lim_{x \to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4}$.

Observation : $x^2 - 2x \to 2^2 - 2 \cdot 2 = 0$. as $x \to 2^-$

$x^2 - 4x + 4 \to 2^2 - 4 \cdot 2 + 4 = 0$ as $x \to 2^-$.

We need to simplify the expression before we compute the limit.

$$\lim_{x \to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4} = \lim_{x \to 2^-} \frac{x(x-2)}{(x-2)^2} = \lim_{x \to 2^-} \frac{x}{x-2}$$

$$\lim_{x\to 2^-} \frac{x^2-2x}{x^2-4x+4} = \lim_{x\to 2^-} \frac{x(x-2)}{(x-2)^2} = \lim_{x\to 2^-} \frac{x}{x-2}$$

Observation : $x \to 2$ as $x \to 2^-$.

$x-2 \to 0$ as $x \to 2^-$

$x \to 2^-$: $x < 2$. so $x-2<0$ negative

$$\frac{x}{x-2} \to \frac{2}{\left(\substack{\text{negative}\\ \text{small} \to 0}\right)} = -\infty$$

Then $$\lim_{x\to 2^-} \frac{x^2-2x}{x^2-4x+4} = \lim_{x\to 2^-} \frac{x}{x-2} = -\infty.$$

#5. Compute the limits.

(a) $$\lim_{x\to -3} \frac{x^2+3x}{x^2-x-12}$$

Observation : $x^2+3x \to (-3)^2+3\cdot(-3)=0$ as $x\to -3$

$x^2-x-12 \to (-3)^2-(-3)-12=0$ as $x\to-3$

$$\lim_{x\to -3} \frac{x^2+3x}{x^2-x-12} = \lim_{x\to -3} \frac{x\cancel{(x+3)}}{(x-4)\cancel{(x+3)}}.$$

$$= \lim_{x\to -3} \frac{x}{x-4}$$

$$= \frac{-3}{-3-4} = \frac{-3}{-7} = \boxed{\frac{3}{7}}.$$

(b) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

Observation : $(2+h)^3-8 \to (2+0)^3-8=0$ as $h\to 0$.

$h \to 0$ as $h \to 0$.

$$\lim_{h\to 0} \frac{(2+h)^3-8}{h} = \lim_{h\to 0} \frac{(2^3+3\cdot 2^2 h + 3\cdot 2\cdot h^2 + h^3)-8}{h}$$

binomial formula: $(a+b)^3 = a^3+3\cdot a^2 b+3a\cdot b^2 + b^3$.

$$= \lim_{h\to 0} \frac{\cancel{8}+12h+6h^2+h^3-\cancel{8}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(12+6h+h^2)}{\cancel{h}}$$

$$= \lim_{h\to 0} (12+6h+h^2) = 12+6\cdot 0+0^2 = \boxed{12}$$

(c). $\lim_{t\to 1} \frac{\sqrt{2-t}-1}{t-1}$.

Observation : $\sqrt{2-t}-1 \to \sqrt{2-1}-1=0$ as $t\to 1$.

$t-1 \to 1-1=0$ as $t\to 1$.

Rewrite / simplify the expression :

$$\lim_{t\to 1} \frac{\sqrt{2-t}-1}{t-1} = \lim_{t\to 1} \frac{(\sqrt{2-t}-1)}{(t-1)} \cdot \frac{(\sqrt{2-t}+1)}{(\sqrt{2-t}+1)}$$

$(a-b)(a+b) = a^2-b^2$.

$\begin{cases} a=\sqrt{2-t} \\ b=1. \end{cases}$

$(\sqrt{2-t})^2 - 1^2$

$a = \sqrt{2-t}$, $b = 1.$

$$= \lim_{t \to 1} \frac{\left(\sqrt{2-t}\right)^2 - 1^2}{(t-1)\left(\sqrt{2-t}+1\right)}.$$

$$= \lim_{t \to 1} \frac{(2-t) - 1}{(t-1)\left(\sqrt{2-t}+1\right)}.$$

$$= \lim_{t \to 1} \frac{\cancel{1-t} \; -1}{\cancel{(t-1)}\left(\sqrt{2-t}+1\right)}$$

$$= \lim_{t \to 1} \frac{-1}{\sqrt{2-t}+1}$$

$$= \frac{-1}{\sqrt{2-1}+1} = \boxed{\frac{-1}{2}}$$

(d) $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right).$

Use squeeze theorem!

Because range of cosine is from -1 to 1

So $-1 \le \cos\left(\frac{2}{x}\right) \le 1$

Then $-x^4 \le \underbrace{x^4 \cos\left(\frac{2}{x}\right)}_{x \to 0} \le x^4.$ $(x^4 \ge 0).$

Since $\lim_{x \to 0} (-x^4) = 0$ and $\lim_{x \to 0} x^4 = 0.$

By squeeze theorem $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0.$

#6. $g(x) = \dfrac{x^2+x-6}{|x-2|}$.

Find $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$. Sketch the graph.

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \frac{x^2+x-6}{|x-2|}$$

$$= \lim_{x\to 2^-} \frac{x^2+x-6}{2-x}$$

$$= \lim_{x\to 2^-} \frac{\cancel{(x-2)}(x+3)}{\cancel{2-x}\ -1}$$

$$= \lim_{x\to 2^-} \frac{x+3}{-1} = \frac{2+3}{-1} = -5$$

As $x\to 2^-$

this means $x<2$

$\Rightarrow$ $x-2<0$.

$\Rightarrow |x-2| = -(x-2)$

$= 2-x$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{x^2+x-6}{|x-2|}$$

$$= \lim_{x\to 2^+} \frac{x^2+x-6}{x-2}$$

$$= \lim_{x\to 2^+} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}}$$

$$= \lim_{x\to 2^+} (x+3) = 2+3 = 5.$$

As $x\to 2^+$

this means $x>2$

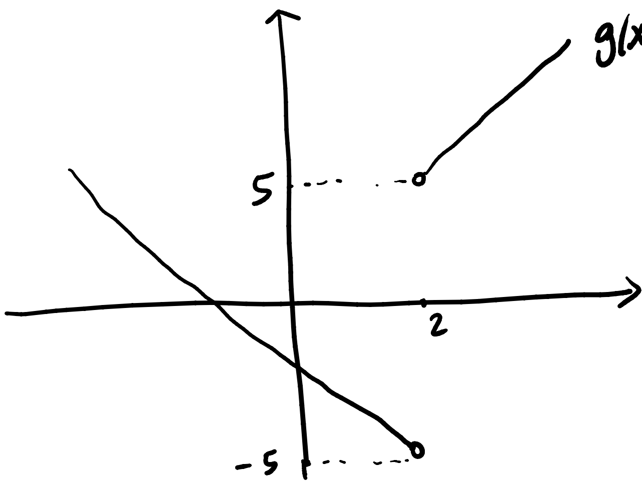
So $x-2>0$.

$\Rightarrow |x-2| = x-2$

Does $\lim_{x\to 2} g(x)$ exist? No, because $\lim_{x\to 2^-} g(x) = -5 \neq 5 = \lim_{x\to 2^+} g(x)$.

Does $\lim_{x\to 2} g(x)$ exist? No, because $\lim_{x\to 2^-} g(x) = -5 \neq 5 = \lim_{x\to 2^+} g(x)$.

From the computation: $g(x) = \dfrac{x+3}{-1}$ $x<2$ (from the left).



$g(x) = x+3$ $x>2$ (from the right).

graph of $g(x)$.