Week in Review 4

Monday, September 26, 2022 4:57 PM

Q 1. Given $\vec{a} = \langle 2,5 \rangle$ $\vec{b} = \langle -1,2 \rangle$. Find $|2\vec{a} + 3\vec{b}|$.

$$2\vec{a} + 3\vec{b} = 2\langle 2,5\rangle + 3\langle -1,2\rangle$$
$$= \langle 4,10\rangle + \langle -3,6\rangle$$
$$= \langle 1,16\rangle$$
$$|2\vec{a} + 3\vec{b}| = \sqrt{1^2 + 16^2} = \sqrt{1+256} = \boxed{\sqrt{257}}.$$

Q 2. Given A(1,-2), B(-4,10). Find vector of length 2 "//" to $\vec{AB}$.

$$\vec{AB} = \langle -4-1, 10-(-2)\rangle = \langle -5, 12\rangle.$$

Unit length : $\frac{\vec{AB}}{|\vec{AB}|}$ $\quad |\vec{AB}| = \sqrt{(-5)^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$$\vec{u} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{\langle -5, 12\rangle}{13} = \langle -\tfrac{5}{13}, \tfrac{12}{13}\rangle$$

$$2\vec{u} = 2\cdot\langle \tfrac{-5}{13}, \tfrac{12}{13}\rangle = \boxed{\langle \tfrac{-10}{13}, \tfrac{24}{13}\rangle}$$

Q3 (a) $\vec{F} = \langle 1,5\rangle$. P(1,0) to Q(7,4).

$$\vec{PQ} = \langle 7-1, 4-0\rangle = \langle 6,4\rangle$$

$$W = \vec{F}\cdot\vec{PQ} = \langle 1,5\rangle\cdot\langle 6,4\rangle$$
$$= 1\cdot 6 + 5\cdot 4$$
$$= 6 + 20 = \boxed{26}$$

(b) $|\vec{F}| = 5$, $\theta = 60^\circ$, $s = 3$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$W = |\vec{F}|\cdot s\cdot\cos\theta$$
$$= 5\cdot 3\cdot\cos 60^\circ$$

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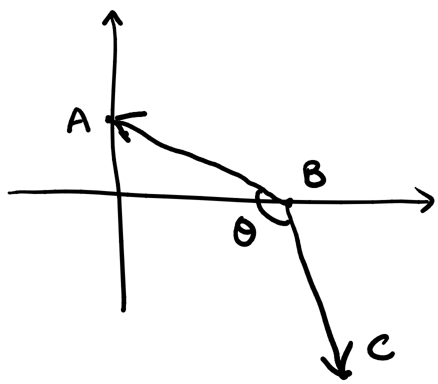
$$= 15 \cdot \frac{1}{2} = \boxed{\frac{15}{2}}$$

angles: 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 0°

Q4. Given $A(0,1)$, $B(2,0)$, $C(3,-4)$.

Find $\angle ABC$.



$$\vec{BA} = \langle 0-2, 1-0 \rangle = \langle -2, 1 \rangle$$

$$\vec{BC} = \langle 3-2, -4-0 \rangle = \langle 1, -4 \rangle$$

So $\vec{BA} \cdot \vec{BC} = |\vec{BA}| \cdot |\vec{BC}| \cos\theta$

$$|\vec{BA}| = \sqrt{(-2)^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$$

$$|\vec{BC}| = \sqrt{1 + (-4)^2} = \sqrt{1+16} = \sqrt{17}$$

$$\vec{BA} \cdot \vec{BC} = \langle -2, 1 \rangle \cdot \langle 1, -4 \rangle = (-2) + (-4) = -6$$

$$\cos\theta = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}|\,|\vec{BC}|} = \frac{-6}{\sqrt{5} \cdot \sqrt{17}} \Rightarrow \angle ABC = \theta = \boxed{\arccos\left(\frac{-6}{\sqrt{5} \cdot \sqrt{17}}\right)}$$

Q5 Given $\vec{a} = \langle 2, 5 \rangle$, $\vec{b} = \langle -1, 2 \rangle$. Find scalar projection from $\vec{a}$ to $\vec{b}$.

$$\text{scal proj}_{\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{\langle 2,5 \rangle \cdot \langle -1, 2 \rangle}{\sqrt{(-1)^2 + 2^2}} = \frac{-2 + 5 \cdot 2}{\sqrt{1+4}} = \boxed{\frac{8}{\sqrt{5}}}$$

Q6.

(a) line passes $(1,-3)$. it is $\perp$ to $\langle 3,-4\rangle$.

we need to find direction $\perp$ to $\langle 3,-4\rangle$.

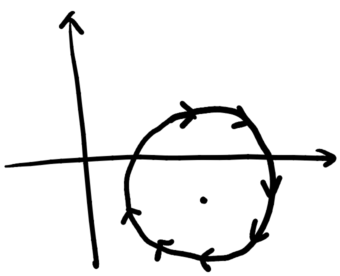
it is $\langle 4, 3\rangle$

So parametric equation is $t\cdot\langle 4,3\rangle + \langle 1,-3\rangle$

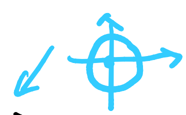
$$= \langle 4t, 3t\rangle + \langle 1,-3\rangle$$

$$= \boxed{\langle 4t+1,\ 3t-3\rangle}.$$

(b) circle, clockwisely. radius 3, center at $(5,-2)$.



unit circle.



counter-clockwise: $\langle \cos(t), \sin(t)\rangle$

clockwise: $\langle \sin(t), \cos(t)\rangle$

$$\begin{cases} x=\cos(t) \\ y=\sin(t) \end{cases}$$

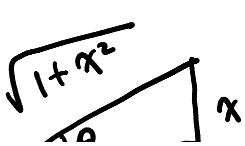
$$3\langle \sin(t), \cos(t)\rangle + \langle 5,-2\rangle$$

$$= \langle 3\sin(t), 3\cos(t)\rangle + \langle 5,-2\rangle$$

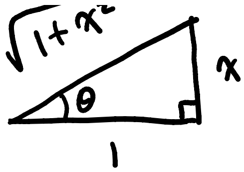
$$= \boxed{\langle 5+3\sin(t),\ -2+3\cos(t)\rangle}.$$

Q7. Simplify $\csc(\underbrace{\arctan(x)}_{\theta})$.

Let $\theta = \arctan(x) \Rightarrow \tan\theta = x$ goal: $\csc\theta = ?$



right triangle: $\csc\theta = \boxed{\dfrac{\sqrt{1+x^2}}{x}}$



right triangle : $\csc\theta = \boxed{\dfrac{\sqrt{1+x^2}}{x}}$

Q8

(a) $\displaystyle\lim_{x\to -4} \frac{x+3}{(x+4)^2} \rightsquigarrow \frac{-4+3}{(-4+4)^2} = \frac{-1}{0^2}$ $\quad \frac{-}{+}$

$= \boxed{-\infty}$

(b) $\displaystyle\lim_{x\to 2^-} \frac{x+1}{x^2+2x-8}$

$\displaystyle = \lim_{x\to 2^-} \frac{x+1}{(x+4)(x-2)} \rightsquigarrow \frac{2+1}{(2+4)\ (2^- -2)} = \frac{3}{6\cdot 0^-}$

$= \boxed{-\infty}$

$x-2<0$ because $x\to 2^-$

$\boxed{\text{tends to 2 from left}}$ $\underline{\underline{<0}}$

(c) $\displaystyle\lim_{x\to 1^-} \frac{x^2+3x-4}{|x-1|}$

$x\to 1^-$: $x-1<0$

$|x-1| = 1-x$

$\displaystyle = \lim_{x\to 1^-} \frac{x^2+3x-4}{1-x}$

$\displaystyle = \lim_{x\to 1^-} \frac{(x+4)\cancel{(x-1)}}{\cancel{1-x}}$ $\quad -1$

$\displaystyle = \lim_{x\to 1^-} (-1)\cdot(x+4) = (-1)\cdot(1+4) = \boxed{-5}$

(d) $$\lim_{x\to-\infty} \frac{5-4x}{\sqrt{9x^2+2x}}$$

" top degree " is x.

divide x on top and on bottom

$$= \lim_{x\to-\infty} \frac{\frac{5}{x}-4}{\sqrt{9x^2+2x}/x}$$

as $x \to -\infty$, x is negative ($x<0$).

$x=-5$

$$\frac{\sqrt{2}}{-5} = -\sqrt{\frac{2}{(-5)^2}}$$

$$= \lim_{x\to-\infty} \frac{\frac{5}{x}-4}{-\sqrt{\frac{9x^2+2x}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{5}{x}-4}{-\sqrt{9+\frac{2}{x}}} = \frac{0-4}{-\sqrt{9+0}} = \frac{-4}{-3} = \boxed{\frac{4}{3}}$$

(e) $$\lim_{x\to-\infty} \frac{2e^{x}-5e^{-x}}{3e^{x}+8e^{-x}}$$

$$\begin{cases} \lim_{x\to-\infty} e^{x} = e^{-\infty} = \frac{1}{e^{\infty}} = 0 & \text{not leading} \\ \lim_{x\to-\infty} e^{-x} = e^{-(-\infty)} = e^{\infty} = \infty & \text{leading} \end{cases}$$

divide leading e^{-x} $\Leftrightarrow$ multiply e^{x}.

$$= \lim_{x\to-\infty} \frac{(2e^{x}-5e^{-x})e^{x}}{(3e^{x}+8e^{-x})e^{x}}$$

$$= \lim_{x\to-\infty} \frac{2e^{2x}-5}{3e^{2x}+8}$$

$$= \frac{0-5}{0+8} = \boxed{-\frac{5}{8}}$$

(f) $$\lim_{x\to\infty} \ln(3x^2+4) - \ln(4x^3+1)$$

formula:

$$\ln A - \ln B = \ln\frac{A}{B}$$

$$= \lim_{x\to\infty} \ln\left(\frac{3x^2+4}{4x^3+1}\right)$$

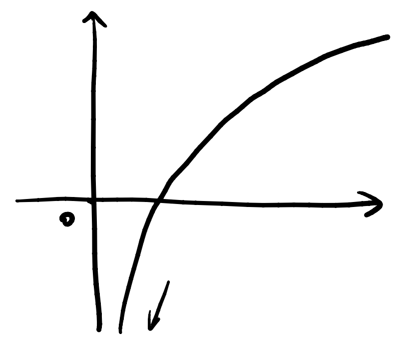
$$= \ln\left(\lim_{x\to\infty} \frac{3x^2+4}{4x^3+1}\right)$$

divide by x^3

$$= \ln\left(\lim_{x\to\infty} \frac{3x^2+4}{4x^3+1}\right)$$

divide by x^3

$$= \ln\left(\lim_{x\to\infty} \frac{\frac{3}{x}+\frac{4}{x^3}}{4+\frac{1}{x^3}}\right).$$



$$= \ln\left(\frac{0+0}{4+0}\right)$$

$$= \ln(0)$$

$$= \boxed{-\infty}$$

Q9. Asymptotes for $f(x) = \dfrac{2x^2+7x+3}{x^2-9}$.

horizontal : $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2+7x+3}{x^2-9}$

$$= \lim_{x\to\infty} \frac{2+\frac{7}{x}+\frac{3}{x^2}}{1-\frac{9}{x^2}} \qquad (\text{divide by } x^2)$$

$$= \frac{2+0+0}{1-0} = 2.$$

similarly : $\displaystyle\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2+7x+3}{x^2-9} = 2.$

$y=2$ is horizontal asymptote.

Vertical : candidate(s) are $x^2-9=0 \Rightarrow x=3,\ x=-3$

look at $x=3$: $\displaystyle\lim_{x\to3^+} \frac{2x^2+7x+3}{x^2-9} = \lim_{x\to3^+} \frac{2x^2+7x+3}{(x-3)(x+3)} \rightsquigarrow \frac{\text{positive}}{0^+\cdot 6}$

$+\infty$.

So $x=3$ is a vertical asymptote ✓

$+\infty$.

So $x=3$ is a vertical asymptote ✓

look at $x=-3$:

$$\lim_{x\to -3^+} \frac{2x^2+7x+3}{x^2-9} = \lim_{x\to -3^+} \frac{(2x+1)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}}$$

$$= \lim_{x\to -3^+} \frac{2x+1}{x-3} = \frac{2\cdot(-3)+1}{-3-3} = \frac{-5}{-6} = \frac{5}{6}$$

finite limit

(same check $x\to -3^-$)
finite limit

So $x=-3$ is NOT a vertical asymptote.

Q 10 Show that there is a solution in $(0,1)$ for $2x^3+16x+3=18$.

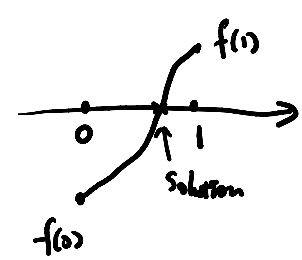
Solution. $\overset{\text{equation}}{\Rightarrow}$ $2x^3+16x-15=0$.

Let $f(x) = 2x^3+16x-15$.

Compute $f(0) = -15 < 0$

$f(1) = 3 > 0$

There is at least one solution



Q 11. (a) $\lim_{x\to 4} \frac{x^2-3x-4}{x^2-2x-8}$

$$= \lim_{x\to 4} \frac{(x-4)(x+1)}{(x-4)(x+2)}$$

$$= \lim_{x\to 4} \frac{x+1}{x+2} = \frac{4+1}{\ldots} = \boxed{\frac{5}{\ldots}}$$

$$= \lim_{x \to 4} \frac{x+1}{x+2} = \frac{4+1}{4+2} = \boxed{\frac{5}{6}}$$

(b) $$\lim_{x \to 5} \frac{\sqrt{x+4} - 3}{x - 5}$$

$$= \lim_{x \to 5} \frac{(\sqrt{x+4} - 3)(\sqrt{x+4} + 3)}{(x-5)(\sqrt{x+4} + 3)}$$

$\boxed{(a-b)(a+b) = a^2 - b^2}$

$$= \lim_{x \to 5} \frac{(x+4) - 3^2}{(x-5)(\sqrt{x+4} + 3)}$$

$$= \lim_{x \to 5} \frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x+4} + 3)}$$

$$= \lim_{x \to 5} \frac{1}{\sqrt{x+4} + 3} = \frac{1}{\sqrt{5+4} + 3} = \boxed{\frac{1}{6}}$$

(c) $$\lim_{x \to 0} x^6 \cos\left(\frac{4}{x}\right)$$

(squeeze theorem): $$-1 \le \cos\left(\frac{4}{x}\right) \le 1$$

$(x^6 \geq 0)$ $$-x^6 \le x^6 \cos\left(\frac{4}{x}\right) \le x^6.$$

We know $\lim_{x \to 0} (-x^6) = 0$ and $\lim_{x \to 0} x^6 = 0.$

$$\lim_{x \to 0} x^6 \cos\left(\frac{4}{x}\right) = 0.$$

Q12. make $f(x) = \begin{cases} x^2 - 5a & x < -1 \\ ax^2 & -1 \le x \le 2 \\ 3ax + b & x > 2 \end{cases}$ continuous everywhere.

Only problem is points at $x=-1$ and $x=2$.

At $x=-1$: the same

$$\left\{ \begin{array}{l} \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2-5a) = 1-5a \\ \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax^2 = a \\ f(1) = a \end{array} \right.$$

$$\Rightarrow \quad 1-5a = a \quad \Rightarrow \quad \boxed{a = \frac{1}{6}.}$$

At $x=2$: the same

$$\left\{ \begin{array}{l} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2 = 4a \\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3ax+b = 6a+b. \\ f(2) = 4a. \end{array} \right.$$

$$\Rightarrow \quad 4a = 6a+b \quad \Rightarrow \quad \boxed{b = -2a = -\frac{1}{3}}$$

Q 13. Use definition to compute $\left(\frac{1}{3x+4}\right)'$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3(x+h)+4} - \frac{1}{3x+4}}{h}$$

$$\frac{1}{3x+3h+4} - \frac{1}{3x+4}$$

$$\frac{3x+4}{(3x+3h+4)(3x+4)} - \frac{3x+3h+4}{(3x+3h+4)(3x+4)}$$

$$= \lim_{h\to 0} \frac{\frac{(3x+4)-(3x+3h+4)}{(3x+3h+4)(3x+4)}}{h}$$

$-3\cancel{h}$

n

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}(3x+3h+4)(3x+4)}$$

$$= \lim_{h \to 0} \frac{-3}{(3x+3h+4)(3x+4)}$$

$$= \frac{-3}{(3x+4)(3x+4)} = \boxed{\frac{-3}{(3x+4)^2}}$$