Week in Review 5

Monday, October 3, 2022 5:02 PM

Q1. (a) $f(x) = x^7 + \sqrt[4]{x} - \frac{5}{x} + \tan(x) - \csc(x).$

$$= x^7 + x^{\frac{1}{4}} - 5x^{-1} + \tan(x) - \csc(x).$$

$$f'(x) = 7x^6 + \frac{1}{4}x^{-\frac{3}{4}} + 5x^{-2} + \sec^2(x) + \csc(x)\cot(x).$$

(b) $h(t) = (2t+5)(3-t). = 6t - 2t^2 + 15 - 5t = -2t^2 + t + 15.$

$$h'(t) = -4t + 1$$

Alternatively, $h'(t) = (2t+5)'(3-t) + (2t+5)(3-t)'$ product rule.

$$= 2\cdot(3-t) + (2t+5)(-1)$$

$$= 6 - 2t - 2t - 5 = -4t + 1.$$

(c) $y = \left(\frac{1}{x^2} - \frac{7}{x^5}\right)(3x-2) = \frac{3}{x} - \frac{2}{x^2} - \frac{21}{x^4} + \frac{14}{x^5} = 3x^{-1} - 2x^{-2} - 21x^{-4} + 14x^{-5}$

$$y' = -3x^{-2} + 4x^{-3} + 84x^{-5} - 70x^{-6}.$$

Alternatively, $y' = \left(\frac{1}{x^2} - \frac{7}{x^5}\right)'(3x-2) + \left(\frac{1}{x^2} - \frac{7}{x^5}\right)(3x-2)'$ $\boxed{\text{product rule}}$

$$= \left(-2x^{-3} + 35x^{-6}\right)(3x-2) + \left(x^{-2} - 7x^{-5}\right)\cdot 3$$

$$= -3x^{-2} + 4x^{-3} + 84x^{-5} - 70x^{-6}.$$

(d) $y = \frac{\sqrt[4]{x^3} + x}{x^2} = \frac{x^{\frac{3}{4}} + x}{x^2} = x^{-\frac{5}{4}} + x^{-1}$

$$\boxed{\frac{x^{\frac{3}{4}}}{x^2} + \frac{x}{x^2} = x^{\frac{3}{4}-2} + x^{1-2}}$$

$$\boxed{y' = -\frac{5}{4}x^{-\frac{9}{4}} - x^{-2}}$$

Alternatively, $y' = \frac{\left(\sqrt[4]{x^3} + x\right)'\cdot x^2 - \left(\sqrt[4]{x^3} + x\right)\cdot\left(x^2\right)'}{\left(x^2\right)^2}$ (quotient rule).

Alternatively, $y' = \frac{\quad\cdots\quad}{(x^2)^2}$ (…).

$$= \frac{(x^{\frac{3}{4}}+x)'\,x^2 - (x^{\frac{3}{4}}+x)\cdot(2x)}{x^4}$$

$$= \boxed{\frac{\left(\frac{3}{4}x^{-\frac{1}{4}}+1\right)x^2 - (x^{\frac{3}{4}}+x)\cdot 2x}{x^4}} = (\text{long simplification}).$$

(e) $p(t) = (2-3t+5t^2)^{50}$ (chain rule).

$$p'(t) = 50\,(2-3t+5t^2)^{49}\cdot(2-3t+5t^2)'$$

$$= \boxed{50\,(2-3t+5t^2)^{49}\cdot(-3+10t)}$$

(f) $f(x) = \dfrac{x^2e^x}{x^2+e^x}$ (quotient rule).

$$f'(x) = \frac{(x^2e^x)'(x^2+e^x) - x^2e^x(x^2+e^x)'}{(x^2+e^x)^2}$$

$$= \frac{(2xe^x+x^2e^x)(x^2+e^x) - x^2e^x(2x+e^x)}{(x^2+e^x)^2}.$$

$$= \frac{\cancel{2x^3e^x} + x^4e^x + 2xe^{2x} + \cancel{x^2e^{2x}} - \cancel{2x^3e^x} - \cancel{x^2e^{2x}}}{(x^2+e^x)^2}$$

$$= \frac{x^4e^x + 2xe^{2x}}{(x^2+e^x)^2}$$

$$= \frac{\quad}{(x^2+e^x)^2}$$

(g) $\quad H(t) = \dfrac{t}{(t^3-7)^5}$ (quotient rule . + chain rule).

$$H'(t) = \frac{(t)'\cdot(t^3-7)^5 - t\cdot\left((t^3-7)^5\right)'}{(t^3-7)^{10}}$$

$$= \frac{(t^3-7)^5 - t\cdot 5(t^3-7)^4\cdot(t^3-7)'}{(t^3-7)^{10}}$$

$$= \frac{(t^3-7)^5 - 5t(t^3-7)^4\cdot 3t^2}{(t^3-7)^{10}}$$

$$= \frac{(t^3-7)^5 - 15t^3(t^3-7)^4}{(t^3-7)^{10}}$$

$$= \frac{t^3-7-15t^3}{(t^3-7)^6} = \frac{-14t^3-7}{(t^3-7)^6}$$

(h) $\quad f(x) = 3^{5x^2-1}$ $\quad$ (recall : $(b^x)' = b^x\cdot\ln b$)

$$f'(x) = \left(3^{5x^2-1}\ln 3\right)\cdot(5x^2-1)'$$

$$= 3^{5x^2-1}\ln 3\cdot 10x$$

$$= \boxed{10\ln 3\ x\cdot 3^{5x^2-1}}$$

(i) $\quad h(t) = e^{t\sin^2(t)}$.

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$$k'(t) = e^{t\sin^2(t)} \cdot \left(t\sin^2(t)\right)'$$

product rule.

$$= e^{t\sin^2(t)}\left(1\cdot \sin^2(t) + t\left(\sin^2(t)\right)'\right).$$

$$= e^{t\sin^2(t)}\left(\sin^2(t) + t\cdot 2\sin(t)\cdot\left(\sin(t)\right)'\right)$$

$$= e^{t\sin^2(t)}\left[\sin^2(t) + 2t\sin(t)\cos(t)\right]$$

(j) $y = \csc\left(\tan\left(\cos(x)\right)\right)$.

$\left(\csc(x)\right)' = -\csc(x)\cot(x)$.

$$y' = -\csc\left(\tan(\cos(x))\right)\cot\left(\tan(\cos(x))\right)\cdot\left(\tan(\cos(x))\right)'$$

$$= -\csc\left(\tan(\cos(x))\right)\cot\left(\tan(\cos(x))\right)\cdot \sec^2\left(\cos(x)\right)\cdot\left(\cos(x)\right)'$$

$$= -\csc\left(\tan(\cos(x))\right)\cot\left(\tan(\cos(x))\right)\cdot \sec^2\left(\cos(x)\right)\cdot\left(-\sin(x)\right)$$

Q2. Given $y = xe^{2x}$. Compute $y^{(2022)}$.

Compute: $y' = (x)'e^{2x} + x\cdot\left(e^{2x}\right)'$

$$= e^{2x} + 2xe^{2x} = e^{2x}(1+2x).$$

$$y'' = \left(e^{2x}\right)'(1+2x) + e^{2x}\cdot(1+2x)'$$

$$y'' = \left(e^{2x}\right)'(1+2x) + e^{2x}\cdot(1+2x)'$$
$$= 2e^{2x}(1+2x) + e^{2x}\cdot 2 = e^{2x}\left(2(1+2x)+2\right)$$
$$= 2e^{2x}(1+2x+1).$$
$$= 2e^{2x}(2+2x).$$

$$y''' = 2\cdot 2e^{2x}(2+2x) + 2e^{2x}\cdot 2$$
$$= 2\cdot 2e^{2x}(2+2x+1) = 2\cdot 2e^{2x}(3+2x).$$

$$y^{(4)} = 2\cdot 2\cdot e^{2x}\cdot 2\,(3+2x) + 2\cdot 2\,e^{2x}\cdot 2$$
$$= 2\cdot 2\cdot 2\, e^{2x}(3+2x+1) = 2\cdot 2\cdot 2\,e^{2x}(4+2x)$$
$$\vdots$$
$$y^{(n)} = 2^{n-1}e^{2x}(n+2x). \qquad (n \geq 1).$$

$n=2022$: $\quad y^{(2022)} = 2^{2021}e^{2x}(2022+2x).$

Q3. Find tangent line eq. of $f(x) = 3x+\sqrt{x}$ at $(4, f(4))$.

Compute $f'(x) = 3 + \left(x^{\frac{1}{2}}\right)' = 3 + \frac{1}{2}x^{-\frac{1}{2}}$

slope : $m = f'(4) = 3 + \frac{1}{2}\cdot 4^{-\frac{1}{2}} = 3 + \frac{1}{2}\cdot\frac{1}{2}$
$$= 3 + \frac{1}{4} = \frac{13}{4}.$$

$$= 3 + \frac{1}{4} = \frac{13}{4}.$$

$$f(4) = 3 \cdot 4 + \sqrt{4} = 3 \cdot 4 + 2 = 14.$$

point-slope : $\quad y - 14 = \frac{13}{4}(x - 4).$

$$\boxed{y = \frac{13}{4}x + 1}$$

Q4. Given $f(x) = \sqrt[3]{x g(x) + 2}$. $g(2) = 3$ and $g'(2) = 1$.

Find normal line eq of $f(x)$ at $(2, f(2))$.

slope of tangent line : $\quad f'(2)$.

$$f'(x) = \left(\left(x g(x) + 2\right)^{\frac{1}{3}}\right)'$$

$$= \frac{1}{3}\left(x g(x) + 2\right)^{-\frac{2}{3}} \cdot \left(x g(x) + 2\right)'$$

$$= \frac{1}{3}\left(x g(x) + 2\right)^{-\frac{2}{3}} \cdot \left((x)' \cdot g(x) + x \cdot g'(x)\right)$$

$$= \frac{1}{3}\left(x g(x) + 2\right)^{-\frac{2}{3}}\left(g(x) + x \cdot g'(x)\right).$$

$$f'(2) = \frac{1}{3}\left(2 g(2) + 2\right)^{-\frac{2}{3}}\left(g(2) + 2 \cdot g'(2)\right).$$

$$= \frac{1}{3} \cdot \left(2 \cdot 3 + 2\right)^{-\frac{2}{3}}\left(3 + 2 \cdot 1\right).$$

$$= \frac{1}{3} \cdot 8^{-\frac{2}{3}} \cdot 5 = \frac{1}{3} \cdot \frac{1}{4} \cdot 5 = \frac{5}{12}.$$

$$8^{-\frac{2}{3}} = \sqrt[3]{8^{-2}} = \left(\sqrt[3]{8}\right)^{-2} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$8^{-\frac{2}{3}} = \sqrt[3]{8^{-2}} = (\sqrt[3]{8})^{-2} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

slope of normal line: $m = -\frac{12}{5}$.

$f(2) = \sqrt[3]{2 \cdot g(2) + 2} = \sqrt[3]{2 \cdot 3 + 2} = \sqrt[3]{8} = 2$

point-slope: $y - 2 = -\frac{12}{5}(x - 2)$.

$$\Rightarrow \boxed{y = -\frac{12}{5}x + \frac{34}{5}}$$

Q5:

$$f(x) = \begin{cases} mx - b & \text{if } x < -1 \\ 5x^2 & \text{if } x \geq -1 \end{cases}$$

Find m and b so f is differentiable everywhere.

Look at $x < -1$: $f(x) = mx - b$ linear function

It's differentiable on $x < -1$.

Look at $x > -1$: $f(x) = 5x^2$ power function.

It's differentiable on $x > -1$.

Only need to check at $x = -1$:

Left-hand side: $\lim\limits_{h \to 0^-} \left. \frac{f(x+h) - f(x)}{h} \right|_{x=-1} = (mx - b)' \Big|_{x=-1}$

$= m$

Right-hand side: $\lim\limits_{h \to 0^+} \left. \frac{f(x+h) - f(x)}{h} \right|_{x=-1} = (5x^2)' \Big|_{x=-1}$

$= 10x \Big| \quad = -10$

$$= 10x\Big|_{x=-1} = -10.$$

left is same as right : $m = -10$.

We also need : $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$.

$$\begin{cases} \text{left} : \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (mx-b) = -m-b \\ \text{right} : \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 5x^2 = 5(-1)^2 = 5 \end{cases}$$

$\Rightarrow \quad -m-b=5 \quad$ (plug in $m=-10$)

$\Rightarrow \quad 10-b=5 \quad \Rightarrow \quad b=5$

So $b=5$ and $m=-10$ will make f differentiable.

Q6: Find a and b so that $y=3x-b$ is tangent to $y=ax^2$ at $x=4$.

Solution. Slope of tangent line must be $y'\Big|_{x=4}$.

$y=3x-b \Rightarrow m=3$ (slope).

On the other hand : $y' = (ax^2)' = 2ax$

$$m = y'\Big|_{x=4} = 2a\cdot 4 = 8a$$

$\Rightarrow \quad 8a = m = 3. \quad \Rightarrow \quad a = \frac{3}{8}$.

$$\Rightarrow \quad 8a = m = 3. \quad \Rightarrow \quad a = \frac{3}{8}.$$

Find out y_0 coordinate :

$$y_0 = ax^2\Big|_{x=4} = a \cdot 4^2 = 16a = 16 \cdot \frac{3}{8} = 6.$$

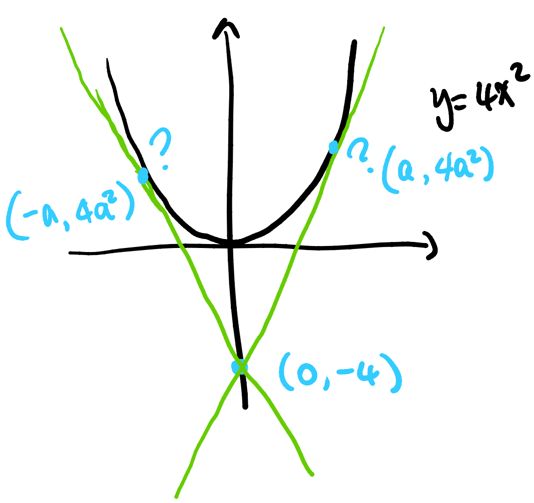
On the other hand: $y = 3x - b$ also pass $(4, 6)$.

So $\quad 6 = 3 \cdot 4 - b$

$$\Rightarrow \quad b = 12 - 6 = 6.$$

$a = \frac{3}{8}$ and $b = 6$.

Q7 : There are two tangent lines to $y = 4x^2$ passing through $(0, -4)$. Find the equations.



<u>Solution</u>. We need to find the slope of tangent lines.

Let $f(x) = 4x^2$, $f'(x) = 8x$.

Assume the point (tangent point) is $(a, 4a^2)$

Then slope is $m = f'(a) = 8a$

Point-slope formula : $y - (-4) = 8a(x - 0)$.

Point-slope formula: $y-(-4) = 8a(x-0)$.

$\Rightarrow \quad y+4 = 8ax$ tangent line.

It passes $(a, 4a^2)$.

$\Rightarrow \quad 4a^2+4 = 8a \cdot a = 8a^2$.

Solve for a: $4a^2 = 4 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

So $(1, 4)$ and $(-1, 4)$ (tangent point)

So tangent line equation is:

$y = 8x-4$ and $y = -8x-4$.