Week in Review 6

Monday, October 10, 2022 5:00 PM

Q 1. Differentiate the functions.

(a) $f(x) = \arctan(3x^2-1)$.

$(\arctan(x))' = \frac{1}{1+x^2}$.

$$f'(x) = \frac{1}{1+(3x^2-1)^2} \cdot (3x^2-1)'$$

$$= \frac{6x}{1+(3x^2-1)^2}$$

(b) $h(t) = \arcsin(t^3e^t)$.

$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$.

$$h'(t) = \frac{1}{\sqrt{1-(t^3e^t)^2}} \cdot (t^3e^t)'$$

$$= \frac{3t^2e^t + t^3e^t}{\sqrt{1-(t^3e^t)^2}} \quad \text{or} \quad \frac{(3t^2+t^3)e^t}{\sqrt{1-t^6e^{2t}}}$$

(c) $g(x) = \ln(4x-6x^2)$.

$(\ln(x))' = \frac{1}{x}$.

$$g'(x) = \frac{1}{4x-6x^2} \cdot (4x-6x^2)'$$

$$= \frac{4-12x}{4x-6x^2} \quad \text{or} \quad \frac{2-6x}{2x-3x^2}$$

(d) $y = \cos(\log_4(x))$

$(\cos(x))' = -\sin(x)$.

$(\log_4(x))' = \frac{1}{x\ln 4}$

$$y' = -\sin(\log_4(x)) \cdot (\log_4(x))'$$

$$y = -\sin(\log_4(x)) \cdot (\log_4(x))'$$

$$= \frac{-\sin(\log_4(x))}{x \ln 4}$$

(e) $$y = (\ln(3x))^{\csc(x)}$$

tower function: $f(x)^{g(x)}$

$\ln(A^B) = B\ln A$

Take ln: $$\ln y = \ln\left(\ln(3x)^{\csc(x)}\right)$$

$$= \csc(x) \ln(\ln(3x))$$

$(\csc(x))' = -\csc(x)\cot(x)$

So $$\frac{1}{y} \cdot y' = \left(\csc(x) \cdot \ln(\ln(3x))\right)'$$

$$= -\csc(x)\cot(x)\ln(\ln(3x)) + \csc(x) \cdot \frac{1}{\ln(3x)} \cdot \frac{1}{3x} \cdot 3$$

$$= -\csc(x)\cot(x)\ln(\ln(3x)) + \frac{\csc(x)}{x\ln(3x)}$$

Solve for

$$y' = y \cdot \left[-\csc(x)\cot(x)\ln(\ln(3x)) + \frac{\csc(x)}{x\ln(3x)}\right]$$

$$= (\ln(3x))^{\csc(x)} \cdot \left[-\csc(x)\cot(x)\ln(\ln(3x)) + \frac{\csc(x)}{x\ln(3x)}\right]$$

(f) $$f(x) = \frac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}$$

Let $$y = f(x) = \frac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}$$

Take ln: $$\ln y = \ln \frac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}$$

$\ln(AB) = \ln A + \ln B$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

Take ln : $\ln y = \ln \dfrac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$\ln(A^B) = B\ln A$

$$= 5\ln(4x+1) + 2\ln(6-5x) - \ln 2 - 9\ln x - (4x^2+7x)$$

$$\frac{1}{y}\cdot y' = \frac{5}{4x+1}\cdot 4 + \frac{2}{6-5x}\cdot(-5) - 0 - \frac{9}{x} - (8x+7).$$

Solve for y' :

$$y' = y\cdot\left[\frac{20}{4x+1} - \frac{10}{6-5x} - \frac{9}{x} - 8x - 7\right].$$

$$= \frac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}\left[\frac{20}{4x+1} - \frac{10}{6-5x} - \frac{9}{x} - 8x - 7\right].$$

Q2. Find tangent line to $x^4 + x^2y^2 + y^3 = 3$ at $(1,1)$.

Differentiate the eq : $4x^3 + 2xy^2 + \underbrace{x^2\cdot 2y\cdot y' + 3y^2\cdot y'} = 0$

Solve for y' : $(x^2\cdot 2y + 3y^2)y' = -(4x^3 + 2xy^2)$

$$y' = \frac{-(4x^3+2xy^2)}{2x^2y+3y^2}$$

Slope at $(1,1)$:

$$y'\Big|_{(1,1)} = \frac{-(4+2)}{2+3} = -\frac{6}{5}$$

Tangent line : $y - 1 = -\frac{6}{5}(x-1)$ or $y = -\frac{6}{5}x + \frac{11}{5}$

Q3. Find tangent line to $xe^y = x - y$ at $(0,0)$.

Differentiate the eq : $e^y + x \cdot e^y \cdot y' = 1 - y'$

Solve for y' : $(xe^y + 1)y' = 1 - e^y$

$$y' = \frac{1-e^y}{xe^y+1}$$

Then slope at $(0,0)$ is $y'\big|_{(0,0)} = \frac{1-e^0}{0+1} = 0$

Tangent line is $y - 0 = 0\cdot(x-0) \Rightarrow y = 0.$

Q4. Find tangent line to $y\sin(2x) = x\cos(2y)$ at $(\frac{\pi}{2}, \frac{\pi}{4})$.

Differentiate eq : $y'\sin(2x) + y\cos(2x)\cdot 2 = \cos(2y) + x\cdot(-\sin(2y))\cdot 2y'$

Solve for y' : $y'\left(\sin(2x) + 2x\sin(2y)\right) = \cos(2y) - 2y\cos(2x).$

$$y' = \frac{\cos(2y) - 2y\cos(2x)}{\sin(2x) + 2x\sin(2y)}.$$

Slope at $(\frac{\pi}{2}, \frac{\pi}{4})$: $y'\big|_{(\frac{\pi}{2},\frac{\pi}{4})} = \frac{\cos(\frac{\pi}{2}) - \frac{\pi}{2}\cos(\pi)}{\sin(\pi) + \pi\sin(\frac{\pi}{2})} = \frac{0 - \frac{\pi}{2}\cdot(-1)}{0 + \pi\cdot 1} = \frac{\frac{\pi}{2}}{\pi} = \frac{1}{2}$

Tangent line is : $y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right) \Rightarrow y = \frac{1}{2}x.$

Q5. Find tangent line to $y = 5x^3\ln(x)$ at $(1,0)$.

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Take derivative : $y' = 15x^2 \cdot \ln(x) + 5x^3 \cdot \frac{1}{x}$

$= 15x^2 \ln(x) + 5x^2$

Slope is (at $(1,0)$) : $y'\big|_{x=1} = 15 \cdot 1 \cdot \ln(1) + 5 \cdot 1^2 = 5$

Tangent line is : $y - 0 = 5(x-1) \Rightarrow y = 5x - 5$

Q 6. $\vec{r}(t) = \langle 2\sin(t) + 2\cos(t),\ 3\cos(t) - 3\sin(t) \rangle$.

(a) Find $\vec{r}'(\frac{2\pi}{3})$.

$\vec{r}'(t) = \langle 2\cos(t) - 2\sin(t),\ -3\sin(t) - 3\cos(t) \rangle$.

$\vec{r}'(\frac{2\pi}{3}) = \langle 2 \cdot \cos(\frac{2\pi}{3}) - 2\sin(\frac{2\pi}{3}),\ -3\sin(\frac{2\pi}{3}) - 3\cos(\frac{2\pi}{3}) \rangle$.

$= \langle 2 \cdot (-\frac{1}{2}) - 2 \cdot \frac{\sqrt{3}}{2},\ -3 \cdot \frac{\sqrt{3}}{2} - 3 \cdot (-\frac{1}{2}) \rangle$.

$= \langle -1 - \sqrt{3},\ \frac{-3\sqrt{3} + 3}{2} \rangle$

(b) Tangent line at $t=0$:

tangent vector $\vec{r}'(0) = \langle 2\cos(0) - 2\sin(0),\ -3\sin(0) - 3\cos(0) \rangle$

$= \langle 2, -3 \rangle$

slope : $m = \frac{-3}{2}$.

Slope: $m = \frac{[illegible]}{2}.$

$$\vec{r}(0) = \langle 2\sin(0) + 2\cos(0), 3\cos(0) - 3\sin(0) \rangle$$

$$= \langle 2, 3 \rangle$$

Tangent line is: $y - 3 = -\frac{3}{2}(x-2)$

(c) Find horizontal tangent line:

$$y'(t) = 0 \Rightarrow -3\sin(t) - 3\cos(t) = 0.$$

$$\Rightarrow \sin(t) = -\cos(t).$$

$$\Rightarrow \tan(t) = -1.$$

$$\Rightarrow t = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}. \quad (0 \le t \le 2\pi).$$

plug in: $(t = \frac{3\pi}{4})$.

$$\vec{r}\left(\frac{3\pi}{4}\right) = \left\langle 2\cdot\sin\left(\frac{3\pi}{4}\right) + 2\cos\left(\frac{3\pi}{4}\right), 3\cos\left(\frac{3\pi}{4}\right) - 3\sin\left(\frac{3\pi}{4}\right) \right\rangle.$$

$$= \left\langle 2\cdot\frac{\sqrt{2}}{2} + 2\cdot\left(-\frac{\sqrt{2}}{2}\right), 3\cdot\left(-\frac{\sqrt{2}}{2}\right) - 3\cdot\frac{\sqrt{2}}{2} \right\rangle.$$

$$= \langle 0, -3\sqrt{2} \rangle.$$

① $y = -3\sqrt{2}$

plug in $(t = \frac{7\pi}{4})$:

$$\vec{r}\left(\frac{7\pi}{4}\right) = \left\langle 2\sin\left(\frac{7\pi}{4}\right) + 2\cos\left(\frac{7\pi}{4}\right), 3\cos\left(\frac{7\pi}{4}\right) - 3\sin\left(\frac{7\pi}{4}\right) \right\rangle.$$

$$= \left\langle -\sqrt{2} + \sqrt{2}, \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} \right\rangle$$

$$= \langle 0, 3\sqrt{2} \rangle.$$

② $y = 3\sqrt{2}.$

(d) Find vertical tangent line:

$x'(t) = 0. \quad \Rightarrow \quad 2\cos(t) - 2\sin(t) = 0.$

$\Rightarrow \quad \sin(t) = \cos(t)$

$\Rightarrow \quad \tan(t) = 1.$

$\Rightarrow \quad t = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}. \quad (0 \le t \le 2\pi).$

plug in $(t=\frac{\pi}{4})$: $\vec{r}(\frac{\pi}{4}) = \langle 2\sin(\frac{\pi}{4}) + 2\cos(\frac{\pi}{4}), 3\cos(\frac{\pi}{4}) - 3\sin(\frac{\pi}{4}) \rangle.$

$= \langle 2 \cdot \frac{\sqrt{2}}{2} + 2 \cdot \frac{\sqrt{2}}{2}, 3 \cdot \frac{\sqrt{2}}{2} - 3 \cdot \frac{\sqrt{2}}{2} \rangle.$

$= \langle 2\sqrt{2}, 0 \rangle.$

① $x = 2\sqrt{2}.$

plug in $(t = 5\frac{\pi}{4})$: $\vec{r}(\frac{5\pi}{4}) = \langle 2\sin(\frac{5\pi}{4}) + 2\cos(\frac{5\pi}{4}), 3\cos(\frac{5\pi}{4}) - 3\sin(\frac{5\pi}{4}) \rangle$

$= \langle 2 \cdot \frac{-\sqrt{2}}{2} + 2 \cdot \frac{-\sqrt{2}}{2}, 3 \cdot \frac{-\sqrt{2}}{2} - 3 \cdot \frac{-\sqrt{2}}{2} \rangle$

$= \langle -2\sqrt{2}, 0 \rangle.$

② $x = -2\sqrt{2}.$

Q7. $\vec{r}(t) = \langle t^4 - 24t + 5, 10t^5 + 1 \rangle.$

(a) Find $\vec{r}'(1).$

Compute $\vec{r}'(t) = \langle 4t^3 - 24, 50t^4 \rangle$

plug in $t=1$: $\vec{r}'(1) = \langle 4 - 24, 50 \cdot 1 \rangle = \langle -20, 50 \rangle.$

(b) Tangent line at $t=0$:

$$\vec{r}'(0) = \langle 4\cdot 0^3 - 24,\ 50\cdot 0^4 \rangle = \langle -24, 0 \rangle.$$

slope : $m = \frac{0}{-24} = 0$.

y coordinate : $\vec{r}(0) = \langle 0^4 - 24\cdot 0 + 5,\ 10\cdot 0^5 + 1 \rangle = \langle 5, 1 \rangle.$

So tangent line is : $y = 1$.

(point-slope : $y - 1 = 0\cdot(x-5) \Rightarrow y = 1$)

(c). Find horizontal tangent line :

$$\vec{r}'(t) = \langle 4t^3 - 24,\ 50t^4 \rangle \Rightarrow x'(t) = 4t^3 - 24,\ y'(t) = 50t^4.$$

horizontal : $y'(t) = 0. \Rightarrow 50t^4 = 0 \Rightarrow t = 0.$

So it reduces to (b) : $y = 1$.

(d) Vertical tangent line :

$$\begin{cases} x'(t) = 4t^3 - 24 \\ y'(t) = 50t^4. \end{cases}$$ Vertical tangent : $x'(t) = 0$.

$$\Rightarrow 4t^3 - 24 = 0 \Rightarrow t^3 = 6. \Rightarrow t = \sqrt[3]{6}.$$

plug in $\vec{r}(\sqrt[3]{6}) = \langle (\sqrt[3]{6})^4 - 24\cdot\sqrt[3]{6} + 5,\ 10\cdot(\sqrt[3]{6})^5 + 1 \rangle.$

$$= \langle \underline{6^{4/3} - 24\cdot 6^{1/3} + 5},\ 10\cdot 6^{5/3} + 1 \rangle$$

$$\boxed{x = 6^{4/3} - 24\cdot 6^{1/3} + 5}$$

Q8. Given $\vec{r}(t) = \langle t, 2t^3 \rangle$. Find angle between velocity and acceleration vectors. at $t=1$.

Compute : $\vec{v}(t) = \vec{r}\,'(t) = \langle 1, 6t^2 \rangle$

$\vec{a}(t) = \vec{v}\,'(t) = \langle 0, 12t \rangle$

plug in $t=1$: $\vec{v}(1) = \langle 1, 6 \rangle$

$\vec{a}(1) = \langle 0, 12 \rangle$.

$$\left(\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| \cdot |\vec{B}|}. \right)$$

$$\cos\theta = \frac{\langle 1,6 \rangle \cdot \langle 0,12 \rangle}{|\langle 1,6 \rangle| \cdot |\langle 0,12 \rangle|} = \frac{72}{\sqrt{1+6^2} \cdot 12} = \frac{6}{\sqrt{37}}$$

$$\theta = \arccos\left(\frac{6}{\sqrt{37}}\right).$$

Q9. Find the point(s) on $y = t^3 - 3t^2 - 12t$, $x = \frac{1}{2}t^2 - t$ so tangent line is parallel to $x = 4t$, $y = 1 - 6t$.

Tangent vector is $x' = t - 1$ and $y' = 3t^2 - 6t - 12$

slope at t is : $\frac{y'}{x'} = \frac{3t^2 - 6t - 12}{t - 1}$.

This slope is the same as $\frac{-6}{4}$. $\left(\text{Given } \begin{cases} y = 1 - 6t \\ x = 4t \end{cases} \right)$.

$$\Rightarrow \quad \frac{3t^2 - 6t - 12}{t - 1} = \frac{-6}{4} = -\frac{3}{2}$$

$$6t^2 - 12t - 24 = -3(t-1).$$

$$6t^2 - 9t - 27 = 0.$$

$$3(2t^2 - 3t - 9) = 0 \Rightarrow 3(2t+3)(t-3) = 0.$$

$$\Rightarrow t = 3 \text{ and } t = -\tfrac{3}{2}.$$

So points are: $\begin{cases} x(3) = \frac{1}{2}\cdot 3^2 - 3 = \frac{3}{2} \\ y(3) = 3^3 - 3\cdot 3^2 - 12\cdot 3 = -36 \end{cases}$ $\left(\frac{3}{2}, -36\right)$.

$\begin{cases} x(-\frac{3}{2}) = \frac{1}{2}\cdot(-\frac{3}{2})^2 - 3 = \frac{21}{8} \\ y(-\frac{3}{2}) = (-\frac{3}{2})^3 - 3(\frac{3}{2})^2 - 12(-\frac{3}{2}) = \frac{63}{8} \end{cases}$ $\left(\frac{21}{8}, \frac{63}{8}\right)$.

Q10. Position function $h(t) = 32t - 16t^2$.

Find v when ball hits the ground.

So $v(t) = h'(t) = 32 - 32t$.

When ball hits the ground: $h(t) = 0$.

Solve for t: $32t - 16t^2 = 0$.

$$16t(2 - t) = 0.$$

$t = 0$ or $t = 2$

Rule out $t = 0$. (ball launch from ground).

Ball hits the ground at $t = 2$.

$$v(2) = 32 - 32\cdot 2 = -32 \text{ ft/s}.$$