Week in Review 8

Monday, October 24, 2022 4:59 PM

Q 1. $x = t^2 + 4t$ and $y = t^2 + 5t$.

$\frac{dy}{dx} = \frac{y'}{x'}$, then compute $x' = 2t + 4$ and $y' = 2t + 5$

<1> vertical tangent line : $x' = 0 \Rightarrow 2t + 4 = 0 \Rightarrow t = -2.$

So plug in: $x(-2) = (-2)^2 + 4(-2) = -4$

$y(-2) = (-2)^2 + 5(-2) = -6$

point : $(-4, -6)$

<2> horizontal tangent line: $y' = 0 \Rightarrow 2t + 5 = 0 \Rightarrow t = -\frac{5}{2}$

So plug in: $x(-\frac{5}{2}) = (-\frac{5}{2})^2 + 4(-\frac{5}{2})$

$= \frac{25}{4} - 10 = -\frac{15}{4}$

$y(-\frac{5}{2}) = (-\frac{5}{2})^2 + 5(-\frac{5}{2}) = \frac{25}{4} - \frac{25}{2} = -\frac{25}{4}.$

point: $(-\frac{15}{4}, -\frac{25}{4}).$

Q 2. $\vec{r}(t) = \langle e^{t^2}, 3t\cos(t) \rangle$

tangent vector : $\vec{r}\,'(t) = \langle e^{t^2} \cdot 2t, 3\cos(t) - 3t\sin(t) \rangle$

@ $t = 0$: $\vec{r}\,'(0) = \langle 0, 3\cos(0) - 0 \rangle$

$= \langle 0, 3 \rangle.$

$|\vec{r}\,'(0)| = |\langle 0, 3 \rangle| = 3$

So unit length : $\frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle 0, 3 \rangle}{3} = \langle 0, 1 \rangle.$

Q 3. Surface area : $A = 4\pi r^2.$

Q3. Surface area: $A = 4\pi r^2$.

Take differential: $dA = 8\pi r\, dr$

max error: $\Delta A \approx |dA| = 8\pi r\, |dr|$

$= 8\pi \cdot 10 \cdot 0.25$

$= 20\pi \quad (in^2)$.

relative error: $\dfrac{\Delta A}{A} \approx \dfrac{20\pi}{4\pi \cdot 10^2} = \dfrac{20\pi}{400\pi} = \dfrac{1}{20} = (5\%)$.

Q4. $H(x) = f\left(g(x^2+4x)\right)$.

$\begin{cases} f'(1) = 2 \\ f'(5) = 0. \end{cases}$

Find $H'(1)$.

$\begin{cases} g(5) = 1 \\ g'(1) = 4 \\ g'(5) = 3 \end{cases}$

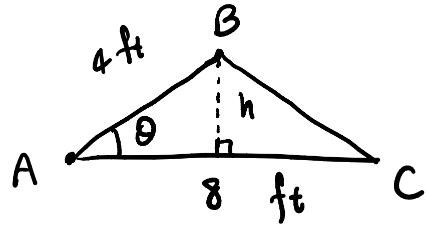
Compute $H'(x) = f'\left(g(x^2+4x)\right) \cdot g'(x^2+4x) \cdot (2x+4)$.

$H'(1) = f'(g(5)) \cdot g'(5) \cdot 6$

$= f'(1) \cdot 3 \cdot 6$

$= 2 \cdot 3 \cdot 6 = \boxed{36}$

Q5



height: $h = 4\sin\theta$

area: $A = \frac{1}{2} \cdot 8 \cdot h$

$= \frac{1}{2} \cdot 8 \cdot 4 \cdot \sin\theta$.

$$= \frac{1}{2} \cdot 8 \cdot 4 \cdot \sin\theta .$$

$$= 16 \sin\theta$$

differentiate equation with respect to t :

$$\frac{dA}{dt} = 16 \cdot \cos\theta \cdot \frac{d\theta}{dt} .$$

angle is decreasing at rate $\frac{\pi}{8}$ rad/s : $\frac{d\theta}{dt} = -\frac{\pi}{8}$.

So $$\left.\frac{dA}{dt}\right|_{\theta=\frac{\pi}{3}} = 16 \cdot \cos\left(\frac{\pi}{3}\right) \cdot \left(-\frac{\pi}{8}\right) .$$

$$= 16 \cdot \frac{1}{2} \cdot \left(-\frac{\pi}{8}\right) = -\pi \quad ft^2/s .$$

Q 6 : Curve $2x^3y - 5y^4 = 11$

Differentiate with respect to x :

$$6x^2y + 2x^3y' - 20y^3 \cdot y' = 0$$

Solve for $y' = \frac{dy}{dx}$:

$$(2x^3 - 20y^3)\, y' = -6x^2y$$

$$y' = \frac{-6x^2y}{2x^3 - 20y^3}$$

$$\left. y' \right|_{(2,1)} = \frac{-6 \cdot 2^2 \cdot 1}{2 \cdot 2^3 - 20 \cdot 1^3} = \frac{-24}{-4} = 6 .$$

point-slope : $y - 1 = 6(x - 2)$.

or $y = 6x - 11$.

Q7. To approximate $\sqrt[3]{10}$

Let $f(x) = \sqrt[3]{x}$. and $a = 8$

Compute $f'(x) = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{x^{2/3}}$

Linear approximation $L(x) = f(a) + f'(a)(x-a)$

$$= \sqrt[3]{8} + \frac{1}{3} \cdot \frac{1}{8^{2/3}} (x-8)$$

$$= 2 + \frac{1}{3} \cdot \frac{1}{4} (x-8).$$

Then $\sqrt[3]{10} = f(10) \approx L(10) = 2 + \frac{1}{3} \cdot \frac{1}{4} (10-8)$

$$= 2 + \frac{1}{6}$$

$$= \frac{13}{6}$$

Q8. $s(t) = t^2 - 4t + 1$. (position function).

Compute $v(t) = s'(t) = 2t - 4$.

Set $v(t) = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$.

It's within 3s. It separates first 3s into:

$[0, 2]$ and $[2, 3]$.

<1> from 0 to 2 : $s(2) - s(0)$

$$= (2^2 - 4 \cdot 2 + 1) - (0^2 - 4 \cdot 0 + 1)$$

$= -4.$

So distance travel is $|S(2) - S(0)| = |-4| = 4.$

<2> from 2 to 3 : $S(3) - S(2)$

$$= (3^2 - 4\cdot 3 + 1) - (2^2 - 4\cdot 2 + 1)$$

$$= (-2) - (-3)$$

$$= 1$$

So distance travel is $|S(3) - S(2)| = 1.$

Combine <1> and <2>. Total distance is $4 + 1 = 5$ ft.

Q9. $g(x) = 2\sin(4x)$

$$g'(x) = 2\cdot 4\cos(4x).$$

$$g''(x) = 2\cdot 4\cdot 4\cdot[-\sin(4x)]$$

$$g'''(x) = 2\cdot 4\cdot 4\cdot 4\,(-1)\cos(4x).$$

$$g^{(4)}(x) = 2\cdot 4\cdot 4\cdot 4\cdot 4\,\sin(4x).$$

$\vdots$

in general
$$g^{(n)}(x) = \begin{cases} 2\cdot 4^n \sin(4x) & \text{if } n=4k \\ 2\cdot 4^n \cos(4x) & \text{if } n=4k+1 \\ 2\cdot 4^n (-1)\sin(4x) & \text{if } n=4k+2 \\ 2\cdot 4^n (-1)\cos(4x) & \text{if } n=4k+3 \end{cases}$$

Now $n = 77$: $(77 = 4k+1)$.

$$g^{(77)}(x) = 2\cdot 4^{77}\cos(4x)$$

Q 10. Population : $P(t) = P_0 e^{kt}$

double every 6 hrs : $P(6) = 2 \cdot P_0$

$$\Rightarrow \quad P_0 e^{6k} = 2 \cdot P_0$$

Solve for k : $e^{6k} = 2$

$$\Rightarrow \quad 6k = \ln 2 \quad \Rightarrow \quad k = \frac{\ln 2}{6}.$$

We see $P(t) = P_0 e^{\frac{t \ln 2}{6}}$

To triple in size : $P(t) = 3P_0$

$$\Rightarrow \quad P_0 e^{\frac{t \ln 2}{6}} = 3P_0$$

solve for t : $e^{\frac{t \ln 2}{6}} = 3$

$$\Rightarrow \quad t\frac{\ln 2}{6} = \ln 3 \quad \Rightarrow \quad t = \frac{6 \ln 3}{\ln 2}$$

Q 11

(a) $y = (3x+1)^{\tan(x)}$ (tower function)

$$\Rightarrow \quad \ln y = \ln (3x+1)^{\tan(x)} = \tan(x) \ln(3x+1).$$

differentiate with respect to x:

$$\frac{1}{y} \cdot y' = \sec^2(x) \ln(3x+1) + \tan(x) \cdot \frac{1}{3x+1} \cdot 3$$

$$= \sec^2(x) \ln(3x+1) + \frac{3\tan(x)}{3x+1}$$

Solve for $\frac{dy}{dx}$:

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y' = y \cdot \left[\sec^2(x) \ln(3x+1) + \frac{3\tan(x)}{3x+1} \right]$$

$$= (3x+1)^{\tan(x)} \left[\sec^2(x) \ln(3x+1) + \frac{3\tan(x)}{3x+1} \right].$$

(b) $$y = \left(\ln(x)\right)^{x^4-7}$$

$$\Rightarrow \quad \ln y = \ln \left(\ln(x)\right)^{x^4-7} = (x^4-7) \ln(\ln(x)).$$

differentiate with respect to x:

$$\frac{1}{y} \cdot y' = 4x^3 \cdot \ln(\ln(x)) + (x^4-7) \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

Solve for y':

$$\frac{dy}{dx} = y' = y \cdot \left[4x^3 \ln(\ln(x)) + \frac{x^4-7}{x \ln(x)} \right]$$

$$= \left(\ln(x)\right)^{x^4-7} \left[4x^3 \ln(\ln(x)) + \frac{x^4-7}{x \ln(x)} \right].$$

(c) $$\sin(xy^3) - \tan(4x) = 2x^3 + 3^{y^2}$$

$$\underbrace{(b^x)' = b^x \cdot \ln b}$$

Differentiate with respect to x:

$$\cos(xy^3)\left(y^3 + \underline{x \cdot 3y^2 \cdot y'}\right) - \sec^2(4x) \cdot 4 = 6x^2 + \underline{3^{y^2} \cdot \ln 3 \cdot 2y \cdot y'}$$

Solve for y'

Solve for y':

$$\left[\cos(xy^3)\cdot 3xy^2 - 3^{y^2}\ln 3\cdot 2y\right]y' = 6x^2 + 4\sec^2(4x) - y^3\cos(xy^3).$$

$$y' = \frac{6x^2 + 4\sec^2(4x) - y^3\cos(xy^3)}{3xy^2\cos(xy^3) - (3^{y^2}\ln 3)\cdot 2y}$$

(d) $y = \arccos(e^{3x})$.

$$\frac{dy}{dx} = y' = -\frac{1}{\sqrt{1-(e^{3x})^2}}\cdot e^{3x}\cdot 3$$

$$= -\frac{3e^{3x}}{\sqrt{1-e^{6x}}}$$

(e) $$y = \ln\left(\frac{e^{3x}(2x+7)^4}{\sqrt[3]{x^2-5}}\right)$$

$$\ln\frac{A}{B} = \ln A - \ln B$$

$$\ln(AB) = \ln A + \ln B$$

$$\ln A^B = B\ln A$$

$$= \ln\left(e^{3x}(2x+7)^4\right) - \ln\sqrt[3]{x^2-5}$$

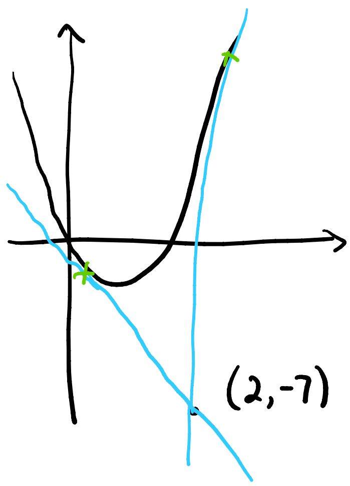
$$= \ln e^{3x} + \ln(2x+7)^4 - \ln(x^2-5)^{\frac{1}{3}}$$

$$= 3x + 4\ln(2x+7) - \tfrac{1}{3}\ln(x^2-5).$$

$$\frac{dy}{dx} = y' = 3 + 4 \cdot \frac{1}{2x+7} \cdot 2 - \frac{1}{3} \cdot \frac{1}{x^2-5} \cdot 2x$$

$$= 3 + \frac{8}{2x+7} - \frac{2x}{3(x^2-5)}$$

Q 12.



Solution. $y = x^2 - x$

compute $y' = 2x - 1$

If tangent line passes the point on the parabola : point $(a, f(a))$.

$f(x) = y = x^2 - x$.

Point : $(a, a^2 - a)$.

slope : $m = 2a - 1$.

Point-slope formula :

$$y - (-7) = (2a-1)(x-2).$$

It passes (a, a^2-a) : $\begin{cases} x = a \\ y = a^2 - a \end{cases}$

$$\Rightarrow \quad a^2 - a + 7 = (2a-1)(a-2)$$

$$= 2a^2 - 5a + 2$$

$$\Rightarrow \quad a^2 - 4a - 5 = 0.$$

$$\Rightarrow \quad (a-5)(a+1) = 0 \Rightarrow a = 5 \text{ and } a = -1.$$

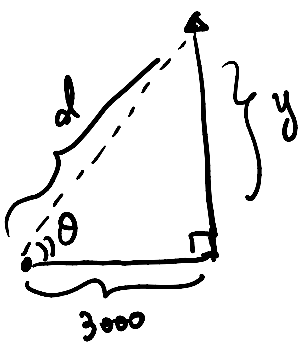
$$\Rightarrow \quad (a-5)(a+1)=0 \quad \Rightarrow a=5 \text{ and } a=-1.$$

Plug back in point-slope formula:

$\langle 1 \rangle \quad y+7 = 9(x-2)$

$\langle 2 \rangle \quad y+7 = -3(x-2).$

Q 13.



(a) Set up: $3000^2 + y^2 = d^2$

Differentiate with respect to t:

$$2y \cdot y' = 2d \cdot d'$$

$$\Rightarrow \quad d' = \frac{2y \cdot y'}{2d}$$

We know $y' = 1500$, $\quad d\Big|_{y=4000} = \sqrt{3000^2 + 4000^2} = 5000$

$$d'\Big|_{y=4000} = \frac{2 \cdot 4000 \cdot 1500}{2 \cdot 5000} = 1200 \ ft/s.$$

(b) Set up: $\tan\theta = \dfrac{y}{3000}$

differentiate with respect to t:

$$\sec^2\theta \cdot \theta' = \frac{y'}{3000}$$

$\left(\sec\theta = \dfrac{1}{\cos\theta}\right).$

$$\Rightarrow \quad \theta' = \cos^2\theta \; y'$$

$$\Rightarrow \quad \theta' = \frac{\cos^2\theta \; y'}{3000}.$$

$\left(\sec\theta = \frac{1}{\cos\theta}\right).$

At that moment: $y = 4000$, $y' = 1500$

$$\cos\theta = \frac{3000}{d} = \frac{3000}{\sqrt{3000^2 + 4000^2}} = \frac{3}{5}$$

$$\theta'\Big|_{y=4000} = \frac{\left(\frac{3}{5}\right)^2 \cdot 1500}{3000} = \frac{9}{25} \cdot \frac{1}{2} = \frac{9}{50} \text{ rad/s}.$$