# Week in Review 9

Monday, October 31, 2022  5:06 PM

Q 1. (a) $f(x) = \frac{7-x}{x+1}$.

Compute $f'(x) = \frac{(7-x)' \cdot (x+1) - (7-x)(x+1)'}{(x+1)^2}$

$$= \frac{(-1)(x+1) - (7-x)\cdot 1}{(x+1)^2}$$

$$= \frac{-x-1-7+x}{(x+1)^2} = \frac{-8}{(x+1)^2}.$$

<1> Set $f'(x)=0 \Rightarrow \frac{-8}{(x+1)^2} = 0$ no solution.

<2> $f'(x)$ doesn't exist $\Rightarrow (x+1)^2 = 0 \Rightarrow x=-1$. (rule out)

Our original function $f(x) = \frac{7-x}{x+1}$, it requires that $x \neq -1$.

$x=-1$ is not in the domain of $f$.

So there is no critical point for $f(x)$.

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(b). $g(x) = (x^3-12x)^{1/3}$

compute $g'(x) = \frac{1}{3}(x^3-12x)^{-\frac{2}{3}} \cdot (3x^2-12)$

$$= \frac{x^2-4}{(x^3-12x)^{2/3}}$$

<1> Set up $g'(x)=0 \Rightarrow x^2-4=0 \Rightarrow x=\pm 2$.

<2> $g'(x)$ doesn't exist $\Rightarrow x^3-12x=0 \Rightarrow x=0, \pm 2\sqrt{3}$.

<2> $g'(x)$ doesn't exist $\Rightarrow$ $x^3-12x=0$ $\Rightarrow$ $x=0, \pm 2\sqrt{3}$.

$(x(x^2-12)=0 \Rightarrow x^2-12=0$ or $x=0)$.

So critical points are $x=\pm 2, 0, \pm 2\sqrt{3}$.

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Q2. (a) $f(x)=\frac{1}{x}$ on $[1,5]$.

(idea: critical pts + endpoints. evaluate and compare).

Compute $f'(x) = -\frac{1}{x^2}$ on $(1,5)$.

<1> Set $f'(x)=0$ no solution

<2> $f'(x)$ doesn't exist $\Rightarrow$ $x=0$, but it's not in $(1,5)$.

No critical points for $f(x)$. on $(1,5)$.

Look at endpoints: $f(1)=1$ and $f(5)=\frac{1}{5}$.

Then $f_{max}=1$ and $f_{min}=\frac{1}{5}$.

(b). $g(x)=-5x^3$ on $[-2,4]$.

Compute $g'(x)=-15x^2$ on $(-2,4)$.

<1> Set up $g'(x)=0$ $\Rightarrow$ $x=0$ is in $(-2,4)$.

<2> $g'(x)$ doesn't exist $\Rightarrow$ no such candidate.

Only critical point for $g(x)$ is $x=0$.

Compare/Evaluate: $g(0)=0$, $g(-2)=-5\cdot(-2)^3=40$

and $g(4)=-5\cdot 4^3=-320$.

and $g(4) = -5 \cdot 4^3 = -320$.

$g_{max} = 40$ and $g_{min} = -320$.

(c) $h(x) = x^2 e^{-x}$ on $[0,4]$.

Compute $h'(x) = 2xe^{-x} + x^2 \cdot (-e^{-x}) = (2x - x^2)e^{-x}$ on $(0,4)$.

<1> Set up $h'(x) = 0 \Rightarrow 2x - x^2 = 0 \Rightarrow x(2-x) = 0$.

$x = 0, 2 \Rightarrow x = 2$. ( $x=0$ endpoint).

<2> $h'(x)$ doesn't exist $\Rightarrow$ no such candidate.

So critical point for $h(x)$ is $x = 2$.

Compare / Evaluate: $h(2) = 4e^{-2}$, $h(0) = 0$, $h(4) = 16e^{-4}$.

$h_{max} = 4e^{-2}$ and $h_{min} = 0$.

$\left( \dfrac{4e^{-2}}{16e^{-4}} = \dfrac{4e^{-2} \cdot e^4}{16e^{-4} \cdot e^4} = \dfrac{4e^2}{16} = \dfrac{e^2}{4} > 1 \quad e \approx 2.7\ldots \right)$.

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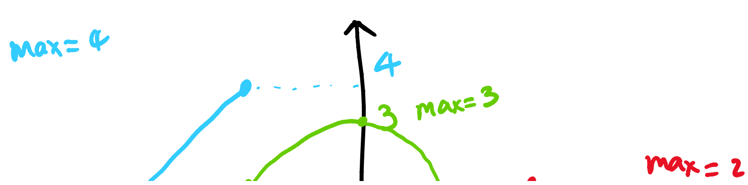
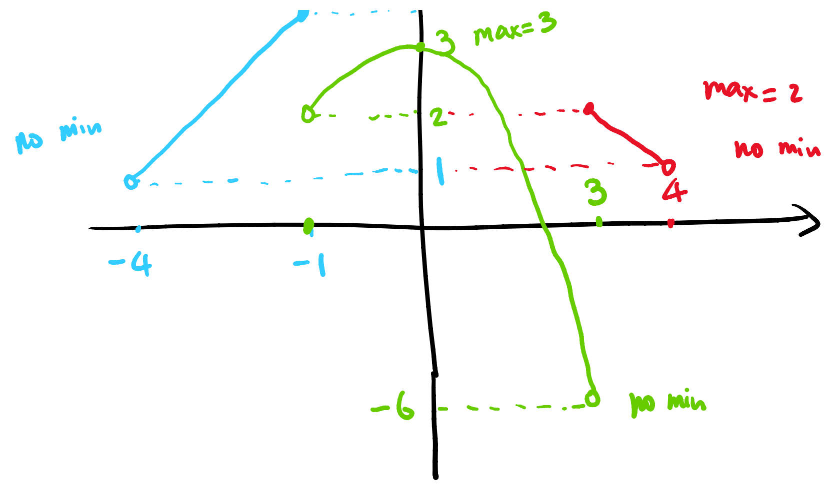
Q3. $f(x) = \begin{cases} x+5 & -4 < x \le -1 \\ 3 - x^2 & -1 < x < 3 \\ 5 - x & 3 \le x < 4 \end{cases}$.

Sketch the grap of each piece :






So $f_{max} = 4$ @ $x = -1$.

$f_{min} = $ DNE

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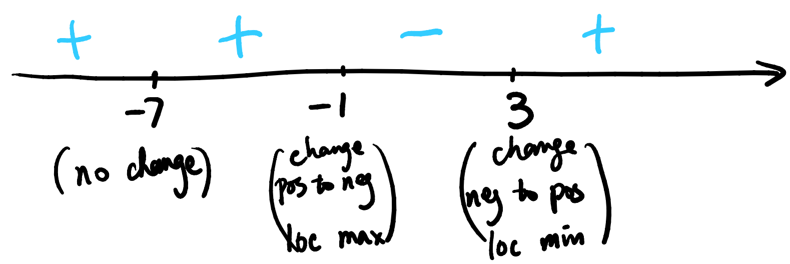
Q4. $f(x)$ and $f'(x) = (x-3)^5(x+1)(x+7)^8$.

classify the local extrema:

We use $1^{st}$-derivative test: consider $f'(x) = (x-3)^5(x+1)(x+7)^8$

signs of $f'$:




So $f$ has loc. max at $x=-1$.

and loc. min at $x=3$.

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Q5 $f'(x) = x(4x-1)^{2/3}$ determine concavity.

(we need $2^{nd}$-derivative).

Compute $f''(x) = (f'(x))' = \left(x(4x-1)^{2/3}\right)'$

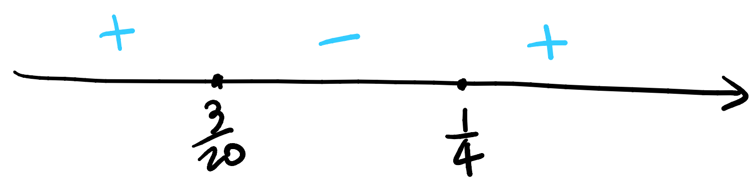
$= (4x-1)^{2/3} + x \cdot \frac{2}{3} \cdot (4x-1)^{-\frac{1}{3}} \cdot 4$

$$= (4x-1) + x \cdot \frac{2}{3} \cdot (4x-1) \cdot 4$$

$$= \frac{(4x-1)}{(4x-1)^{1/3}} + \frac{\frac{8}{3}x}{(4x-1)^{1/3}}$$

$$= \frac{3(4x-1)+8x}{3(4x-1)^{1/3}} = \frac{20x-3}{3(4x-1)^{1/3}}$$

Check signs of $f''$:




$f''>0 \Rightarrow f$ is concave up : on $(-\infty, \frac{3}{20})$ and $(\frac{1}{4}, \infty)$

So inflection points are $x = \frac{3}{20}$ and $x = \frac{1}{4}$.

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Q6. $f(x) = x^2 \ln\left(\frac{x}{4}\right)$.

Compute $f'(x) = 2x\ln\left(\frac{x}{4}\right) + x^2 \cdot \frac{1}{\frac{x}{4}} \cdot \frac{1}{4}$

$$= 2x\ln\left(\frac{x}{4}\right) + x$$

and $f''(x) = 2 \cdot \ln\left(\frac{x}{4}\right) + 2x \cdot \frac{1}{\frac{x}{4}} \cdot \frac{1}{4} + 1$

$$= 2 \cdot \ln\left(\frac{x}{4}\right) + 2 + 1 = 2\ln\left(\frac{x}{4}\right) + 3$$

$2\ln\left(\frac{x}{4}\right) + 3 = 0$

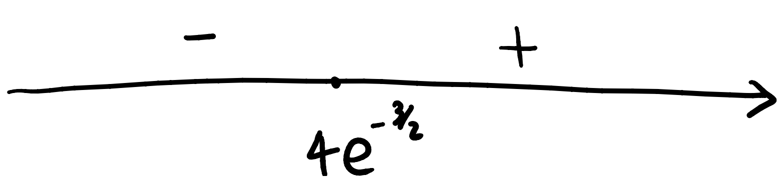
$2\ln\left(\frac{x}{4}\right) = -3$

$\ln\left(\frac{x}{4}\right) = -\frac{3}{2}$

$\frac{x}{4} = e^{-3/2}$

$x = 4 \cdot e^{-3/2}$

Check signs of $f''$:



$4e^{-3/2}$

$\frac{x}{4} = e^{-3/2}$
$x = 4 \cdot e^{-3/2}$

So $f$ is concave up when $4e^{-3/2} < x < \infty$
( on $(4e^{-3/2}, \infty)$ ).

inflection point is $x = 4e^{-3/2}$.

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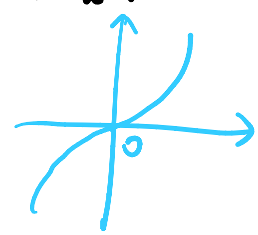
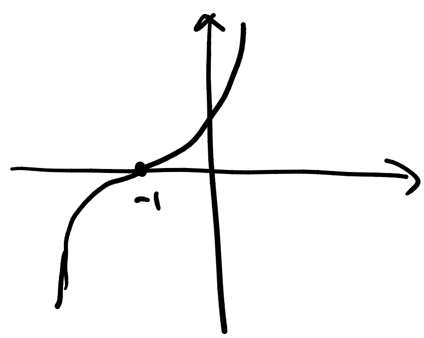
Q 7. Find a function continuous, $x=-1$ critical point but $f$ has no local extrema.

Think of $y = x^3$ at $x=0$




So shift the function to left by 1 unit

$\Rightarrow$ $f(x) = (x+1)^3$ at $x = -1$.

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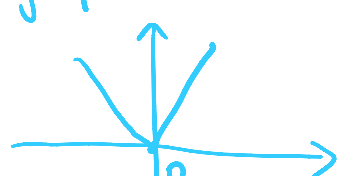
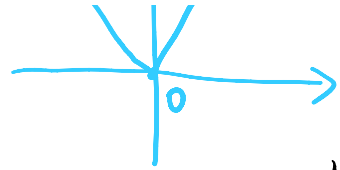
Q8. Find $f(x)$. $x=3$ local. min, but $f'(3)$ not exist.

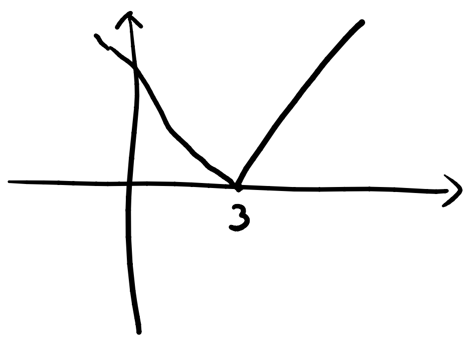
Think of $y = |x|$ at $x=0$






Shift it by 3 units to right :

$f(x) = |x-3|$. at $x=3$

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Q9. $f(x) = x\sin(x) + \cos(x)$ on $[0, 2\pi]$.

( $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ ).

Verify : $f$ is continuous on $[0, 2\pi]$
and $f$ is differentiable on $(0, 2\pi)$. ✓

Compute : $f'(x) = \sin(x) + x\cos(x) - \sin(x)$
$= x\cos(x)$.

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2\pi)-f(0)}{2\pi - 0} = \frac{2\pi\sin(2\pi) + \cos(2\pi) - (0 + \cos(0))}{2\pi}$$

$$= \frac{1-1}{2\pi} = 0$$

Set up $f'(c) = \dfrac{f(b)-f(a)}{b-a} = 0$ . on $(0, 2\pi)$.

Set up $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$ . on $(0, 2\pi)$.

$\Rightarrow \quad c\cos(c) = 0 \qquad (c \neq 0, \quad 0 < c < 2\pi)$.

$\Rightarrow \quad \cos(c) = 0$.

$\Rightarrow \quad c = \frac{\pi}{2}$ and $c = \frac{3\pi}{2}$.

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Q10. $P = P(t)$ population. $P$ is continuous on $[0, 20]$
is differentiable on $(0, 20)$.

$P(0) = 50$ , $1 \leq P'(t) \leq 5$.

By MVT: there is a $c$ in $(0, 20)$
so that: $P'(c) = \frac{P(20) - P(0)}{20 - 0}$

Because $1 \leq P'(t) \leq 5 \Rightarrow 1 \leq P'(c) \leq 5$

$\Rightarrow \quad 1 \leq \frac{P(20) - P(0)}{20} \leq 5$

$\Rightarrow \quad 20 \leq P(20) - P(0) \leq 100$

$\Rightarrow \quad 20 + P(0) \leq P(20) \leq 100 + P(0)$

Since $P(0) = 50 \Rightarrow$

$20 + 50 \leq P(20) \leq 100 + 50$

So $P(20)_{max} = 150$ and $P(20)_{min} = 70$.