WIR 10

Week-in-Review 10: 4.1-4.4
(Maxima, Minima, Mean value, Shapes of curves, L'Hospitals.)

Extreme Value theorem

Problem 1. Find the Absolute Maxima and the Absolute Minina for the given functions:

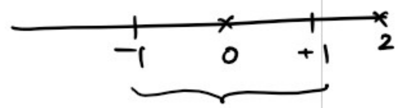
a) $f(x) = \underbrace{x^3 - 3x^2 - 1}$ on the interval $[-1, 1]$. f(x) is continuous in $[-1,1]$

1) find critical numbers (CN) for f(x)

$$f'(x) = \underbrace{3x^2 - 6x} = 0$$

$$\underbrace{3x}\,\underbrace{(x-2)} = 0$$

$$x = 0 \qquad \cancel{x = 2}$$



CN $\quad f(x=0) = -1 \rightarrow$ Absolute maxima. (largest y value)

end points

$$f(x=-1) = (-1)^3 - 3(-1)^2 - 1 = -1-3-1 = -5 \rightarrow \text{Abs minima. (smallest y value)}$$

$$f(x=1) = (1)^3 - 3(1)^2 - 1 = 1-3-1 = -3$$

b) $f(x) = \sqrt{6x - x^2}$ in its allowed Domain.

in interval $[0,6]$

find D for $f(x)$

$$f(x) = \sqrt{6x - x^2}$$

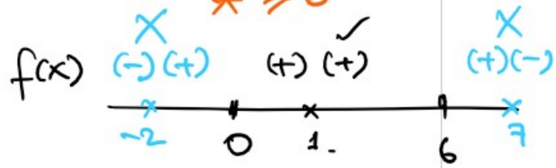
If $\sqrt[n]{*}$, $n =$ even

$* \geq 0$

$$6x - x^2 \geq 0$$

$$x(6 - x) \geq 0$$

@ $x = 0$ @ $x = 6$



D of $f(x)$: $[0,6]$.

find CNs

$$f'(x) = \frac{1}{2\sqrt{6x-x^2}} \frac{d}{dx}(6x - x^2)$$

$$= \frac{6 - 2x}{2\sqrt{6x - x^2}}$$

$f'(x) = 0$ $\boxed{@\ x = 3}$

$f'(x)$ DNE @ $x = 0$, $x = 6$

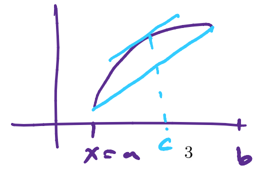
CN $f(x=3) = \sqrt{6(3) - (3)^2} = \sqrt{18 - 9} = \sqrt{9} = 3$

Absolute maxima.

end points
$$\begin{cases} f(x=0) = \sqrt{6(0) - 0^2} = 0 \\ f(x=6) = \sqrt{6(6) - 6^2} = 0 \end{cases}$$

Absolute minima is $f(x) = 0$.

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$



Problem 2. Use the Mean Value Theorem to find the number c that satisfies the theorem on the given interval.

a) $f(x) = 2x^2 - 3x + 1$ on the interval $[0, 2]$.

$$f'(x) = 4x - 3$$

$$f'(c) = 4c - 3 = \frac{f(2) - f(0)}{2 - 0}$$

$$= \frac{2(2^2) - 3(2) + 1 - (0 - 0 + 1)}{2}$$

$$= \frac{8 - 6 + 1 - 1}{2} = \frac{2}{2} = 1$$

$4c - 3 = 1$

$4c = 4$

$c = 1$ Ans.

b) $f(x) = \ln x$ on the interval $[1, 4]$.

$$f'(x) = \frac{1}{x}$$

$$f'(c) = \frac{1}{c} = \frac{f(4) - f(1)}{4 - 1} = \frac{\ln(4) - \cancel{\ln(1)}^{0}}{3}$$

$$\frac{1}{c} = \frac{\ln(4)}{3}$$

$$\therefore \boxed{c = \frac{3}{\ln(4)}}$$ Ans.

$f'(x) = 0$ /
$f'(x)$ DNE

$f'(x) > 0, \quad f(x) \nearrow$

$f'(x) < 0, \quad f(x) \searrow$

Problem 3. Find the critical numbers and then find the intervals where the following functions are increasing or decreasing. Identify each critical number as a maxima, minima or neither.

a) $f(x) = \dfrac{x+3}{5-x}$ $\leftarrow$ D: $x \neq 5$

$$f'(x) = \frac{(5-x)(1) - (x+3)(-1)}{(5-x)^2} = \frac{5-x+x+3}{(5-x)^2}$$

$$\boxed{f'(x) = \frac{8}{(5-x)^2}}$$

$\to$ $f'(x)$ DNE @ $x = 5$ $\to$ $f(x)$ has no max & no min

$f'(x)$: $+$ at 0, $x=5$, $+$ at 10: $\dfrac{8}{(5-10)^2} = \dfrac{8}{25}$

$(-\infty, 5) \to f(x)$ is increasing

$(5, \infty) \to f(x)$ is increasing

$f(x)$ is never decreasing

There is a vertical asymptote @ $x = 5$

HA: $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \infty} \dfrac{x+3}{5-x} \sim \dfrac{x}{(-x)} = -1$

$\lim\limits_{x \to (-\infty)} \dfrac{x+3}{5-x} \sim \dfrac{x}{(-x)} = -1$

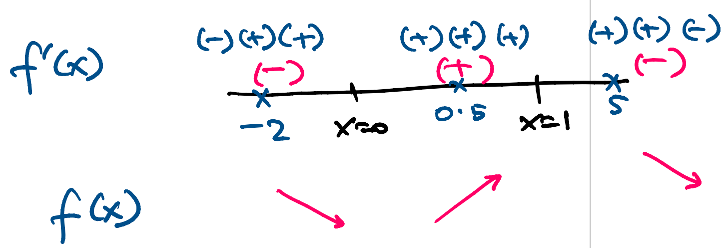
b) $f(x) = \underbrace{x^2}\underbrace{e^{-2x}}$

$$f'(x) = (2x)(e^{-2x}) + (x^2)(-2e^{-2x})$$

$$f'(x) = 2\underbrace{x}_{x=0}\,\underbrace{e^{-2x}}_{\text{never zero!}}\,\underbrace{(1-x)}_{x=1} = 0$$

$x=0 \rightarrow$ local minima.

$x=1 \rightarrow$ local maxima.



f(x) is increasing in $(0,1)$

f(x) is decreasing in $(-\infty,0)\cup(1,\infty)$

$+ \; +$ $\quad - \; -$

concavity

$f''(x) = 0 \,/\, f''(x)$ DNE

$f''(x) > 0 \rightarrow f(x) \cup$

$f''(x) < 0 \rightarrow f(x) \cap$

Problem 4. Identify the inflection points and find the intervals where the following functions are concave up or concave down.

a) $f(x) = \ln(x^2 + 6x - 13)$

D: all x

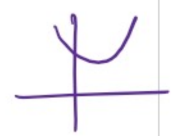
$\log_b(*)$
$* \geq 0$

$f'(x) = \dfrac{2x+6}{x^2+6x+13} \rightarrow$ CN: $f'(x)=0$ @ $x=-3$

$x^2+6x+13=0$

$x = \dfrac{-6 \pm \sqrt{36-52}}{2}$

has no roots



$$f''(x) = \frac{(x^2+6x+13)(2) - (2x+6)(2x+6)}{(x^2+6x+13)^2}$$

$2x^2 + 12x + 26 - (4x^2 + 12x + 12x + 36)$

$-2x^2 - 12x - 10$

$-2(x^2+6x+5) = (-2)(x+5)(x+1)$

$$f''(x) = \frac{(-2)(x+5)(x+1)}{(x^2+6x+13)^2}$$

$x=-5$, $x=-1$ $\quad f''(x)=0$.

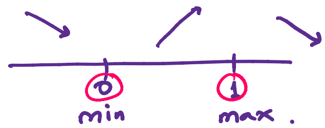
Inflection pts @ $x = -5, -1$

$f''(x)$: $(-)(-)(-)$ $(-)$ at -6; $(-)(+)(-)$ $(+)$ at -3; $(-)(+)(+)$ $(-)$ at 0; marks at -5 and -1

$f(x)$: $\cap$ $\quad \cup$ $\quad \cap$

$f(x)$ is concave up $(-5, -1)$

$f(x)$ is concave down $(-\infty, -5) \cup (-1, \infty)$



b) $f(x) = x^2 e^{-2x}$

$$f'(x) = 2x\,e^{-2x} - 2x^2\,e^{-2x}$$

$$f''(x) = 2e^{-2x} - 4x\,e^{-2x} - \left(4x\,e^{-2x} - 4x^2 e^{-2x}\right)$$

$$= 2e^{-2x} - 4x\,e^{-2x} - 4x\,e^{-2x} + 4x^2 e^{-2x}$$

$$= 2e^{-2x}\left(1 - 2x - 2x + 2x^2\right)$$

$$f''(x) = \underbrace{2e^{-2x}}_{\text{never zero!}}\left(\underbrace{1 - 4x + 2x^2}\right) = 0$$

$$2x^2 - 4x + 1 = 0$$

$$x = \frac{(+4) \pm \sqrt{16 - 8}}{4}$$

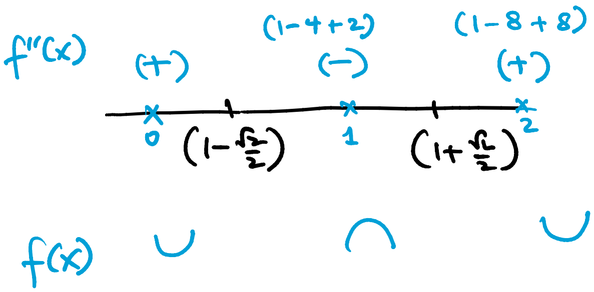
$$= \frac{4 \pm \sqrt{8}}{4}$$

$$= 1 \pm \frac{2\sqrt{2}}{4} = 1 \pm \frac{\sqrt{2}}{2}$$

$1 - \frac{\sqrt{2}}{2}$ ~ 0.29 $1 + \frac{\sqrt{2}}{2}$ ~ 1.71

Inflection points @

$x = 1 - \frac{\sqrt{2}}{2}$, $x = 1 + \frac{\sqrt{2}}{2}$



$f(x)$ is concave up in $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right) \cup \left(1 + \frac{\sqrt{2}}{2}, \infty\right)$

$f(x)$ is concave down in $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$

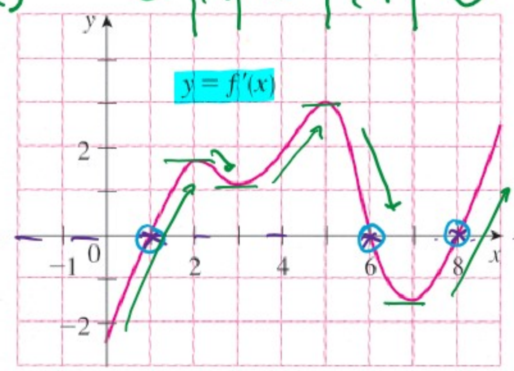
$f(x)$ is $\cup$ if $f''(x) > 0$ ie $f'(x) \nearrow$

$f(x)$ is $\cap$ if $f''(x) < 0$ ie $f'(x) \searrow$

Problem 5. Given the following graphs, identify all the local extrema and inflection points.

$f(x)$ $\quad \cup \mid \cap \mid \cup \mid \cap \mid \cup$

$f'(x)>0$ $\uparrow$ $f'(x)=0$ $\downarrow$ $f'(x)<0$



$\longrightarrow$ graph of 1st derivative $f'(x)$

CNs: $x = 1, 6, 8$

min $\quad$ max $\quad$ min

@ $x = 2, 3, 5, 7$

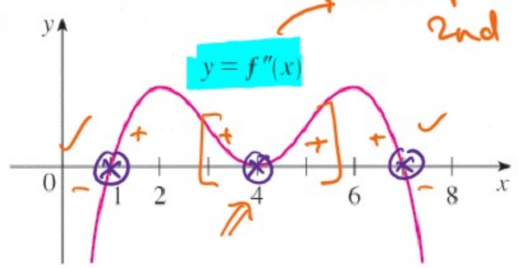
Inflection pts: $f''(x)=0$

$f''(x) = \underbrace{\frac{d}{dx} f'(x) = 0}_{\text{horizontal tangent line for } f'(x)}$

a) .

$f(x)$: $\searrow \mid \nearrow \mid \searrow \mid \nearrow$

$f''(x)>0$ $\uparrow$ $f''(x)=0$ $\downarrow$ $f''(x)<0$



Graph of 2nd derivative.

b) .

$f(x)$ $\quad \cap \mid \cup \mid \cap$

Inflection pts: $f''(x)=0$.

@ $x = 1, \cancel{4}, 7$.

$f''(x)$ does not change sign around $x = 4$.

CNs: $f'(x)=0$

Graph of $f''(x)$ cannot give me any information about CNs., local max/local min, intervals where $f(x)$ increases or decreases.

$$\lim_{x\to a} \frac{f(x)}{g(x)} \xrightarrow{L'H} \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{if } \frac{0}{0}, \frac{\infty}{\infty} \leftarrow \text{Indeterminate form.}$$

Problem 6. Find the following limits. You may use L'Hospital's rule, if applicable.

a) $\lim_{x\to\infty} \frac{\ln(5x)}{x^2}$ $\quad \xrightarrow{x\to\infty} \frac{\ln(\infty)}{(\infty)^2} \sim \frac{\infty}{\infty}$ ✓

Problem 6. Find the following limits. You may use L'Hospital's rule, if applicable.

a) $\lim_{x\to\infty} \frac{\ln(5x)}{x^2}$ $\xrightarrow{x\to\infty}$ $\frac{\ln(\infty)}{(\infty)^2} \sim \frac{\infty}{\infty}$ ✓

$\xrightarrow{L'H}$ $\frac{\frac{1}{\cancel{5}x}\cdot(\cancel{5})}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} \to \frac{1}{\infty^2} = \boxed{0}$

$x \to \left(\frac{\pi}{2}\right)^-$

b) $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$ $\xrightarrow{x=\pi/2}$ $\frac{1}{\cos(\pi/2)} - \tan(\pi/2) = \frac{1}{0} - \infty$

$\sim (\infty - \infty)$

Indeterminate difference.

$= \frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}$

$= \frac{1-\sin(x)}{\cos(x)} \xrightarrow{x=\pi/2} \frac{1-1}{0} \sim \frac{0}{0}$ ✓

$\xrightarrow{L'H}$ $\frac{-\cos(x)}{-\sin(x)}$ $\xrightarrow{x=\pi/2}$ $\frac{\cos(\pi/2)}{\sin(\pi/2)} = \frac{0}{1} = \boxed{0}$

$\frac{d}{dx}(1-\sin x)$

$\frac{d}{dx}(\cos x)$

c) $\lim_{x\to 0} \frac{1-\cos x}{2x^2}$ $\xrightarrow{x=0}$ $\frac{1-\cancel{\cos}^{1}(0)}{2(0)} \sim \frac{0}{0}$ ✓

$\xrightarrow{L'H}$ $\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(2x^2)} = \boxed{\frac{\sin(x)}{4x}} \xrightarrow{x=0} \frac{\sin(0)}{0} \sim \frac{0}{0}$ ✓

$\xrightarrow{L'H}$ $\frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(4x)} = \frac{\cos(x)}{4} \xrightarrow{x=0}$

$\frac{\cos(0)}{4} = \frac{1}{4}$

d) $\underbrace{\lim_{x\to\infty}(xe^{1/x} - x)}$ $\xrightarrow{x\to\infty}$ $(\infty)e^{\frac{1}{\infty}} - \infty = \underbrace{(\infty)1 - \infty}_{\text{Indeterminate difference!}}$

$xe^{\frac{1}{x}} - x$

$\underbrace{(x)}\underbrace{(e^{\frac{1}{x}} - 1)} \xrightarrow{x\to\infty} (\infty)(e^0 - 1) \sim \underbrace{[(\infty)(0)]}_{\text{Indet. product!}}$

$(a)(b) \nearrow \frac{a}{(\frac{1}{b})}$, $(a)(b) \searrow \frac{b}{(\frac{1}{a})}$

$\lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{(\frac{1}{x})} \xrightarrow{x\to\infty} \frac{0}{0}$ ✓

$\xrightarrow{L'H}$ $\frac{(e^{\frac{1}{x}})\cdot\cancel{(-\frac{1}{x^2})}}{\cancel{(-\frac{1}{x^2})}} = \lim_{x\to\infty} e^{\frac{1}{x}}$

$= e^{\frac{1}{\infty}} = e^0 = 1.$

e) $\lim_{x\to\infty}(\underbrace{1+e^{2x}}_{\text{base}})^{1/x}$ exp $\xrightarrow{x\to\infty}$ $(1+e^{\infty})^{(\frac{1}{\infty})} \sim \infty^0$

Indet. power!

e) $\lim\limits_{x\to\infty} (\underbrace{1+e^{2x}}_{\text{base}})^{1/x}$ $\xrightarrow{x}$ $(1+\ldots)$ 1 Indet. power!

Ans.

$$y = \lim_{x\to\infty} (1+e^{2x})^{\frac{1}{x}}$$

$$\boxed{\ln y} = \ln \lim_{x\to\infty} (1+e^{2x})^{\frac{1}{x}}.$$

$$= \lim_{x\to\infty} \left(\frac{1}{x}\right)\ln(1+e^{2x}) \xrightarrow{x\to\infty} \underbrace{(0)(\cancel{1+}\infty)}_{\text{Indet. product}}$$

$$= \lim_{x\to\infty} \frac{\ln(1+e^{2x})}{x} \sim \frac{\infty}{\infty}\checkmark.$$

$$\xrightarrow{L'H} \lim_{x\to\infty} \frac{\left(\frac{1}{1+e^{2x}}\right)(2e^{2x})}{1} = \boxed{\frac{2e^{2x}}{1+e^{2x}}} \sim \frac{\infty}{\infty}\checkmark$$

$$\xrightarrow{L'H} \frac{2\cancel{4}\cancel{e^{2x}}}{\cancel{2}\cancel{e^{2x}}} = 2 = \ln y$$

$$y = e^2$$

f) $\lim\limits_{x\to\infty} (1+\frac{3}{x}+\frac{5}{x^2})^x$

$$\xrightarrow{x\to\infty} (1)^{\infty}$$

$$y = \lim_{x\to\infty} \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$$

$$\ln y = \lim_{x\to\infty} \underbrace{x}\,\underbrace{\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)} \sim \underbrace{(\infty)(\cancel{1}\,0)}_{\text{Indet. product}}$$

$$= \frac{\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{\left(\frac{1}{x}\right)} \sim \frac{0}{0}\checkmark$$

$$\xrightarrow{L'H} \frac{\left(\frac{1}{1+\frac{3}{x}+\frac{5}{x^2}}\right)\cdot\left(0-\frac{3}{x^2}-\frac{5(2)}{x^3}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \frac{\left(-\frac{3}{x^2}-\frac{10}{x^3}\right)(-x^2)}{\left(1+\frac{3}{x}+\frac{5}{x^2}\right)} = \lim_{x\to\infty} \frac{3+\cancel{\frac{10}{x}}^{0}}{\left(1+\cancel{\frac{3}{x}}^{0}+\cancel{\frac{5}{x^2}}^{0}\right)} = 3$$

$$\ln y = 3$$

$$y = e^3 \quad \text{Ans.}$$