WEEK-IN-REVIEW 10: 4.7, 4.9 AND 5.1
(OPTIMIZATION, ANTIDERIVATIVES, RIEMANN SUMS.)

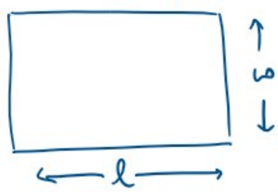
Optimization $\to$ max, $\to$ min } CNs $f(x)$ $\Rightarrow f'(x) = 0$. 1 var. only

a) Q: $\ell = ?$ $w = ?$

Problem 1. Find the dimensions of a rectangle of area 225 square centimeters that has the smallest perimeter. What is the perimeter?

b) $P = ?$

minimize perimeter P



$$\boxed{A = 225 = \ell \cdot w}$$

$$\ell = \frac{225}{w}$$

f(x) maxima.

minima.

$f''(CN) < 0$

$f''(CN) > 0$

$$P = 2\ell + 2w \leftarrow \text{2 variables}$$

$$P = 2\left(\frac{225}{w}\right) + 2w = \frac{450}{w} + 2w \leftarrow \text{only have 1 variable}$$

$$\frac{dP}{dw} \leftarrow P' = \underbrace{-\frac{450}{w^2} + 2}= 0$$

$$2 = \frac{450}{w^2} \Rightarrow w^2 = \frac{450}{2} = 225$$

$$w = \sqrt{225} = \pm 15 \leftarrow \text{CN.}$$

proove CN $\to$ min

$$P'' = -450 \cdot \left(-\frac{2}{w^3}\right) = \frac{2(450)}{w^3} > 0$$

$\therefore$ CN $\to$ min

Ans: Smallest perimeter

a) $w = 15$ cm ; $\ell = \frac{225}{w} = 15$ cm ; b) Perimeter $= 2(15) + 2(15)$
$= 60$ cm

$V = (\pi r^2)h = 192\pi$

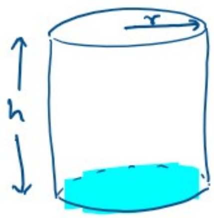
$h = \frac{192\cancel{\pi}}{\cancel{\pi} r^2}$

$h = \frac{192}{r^2}$

Problem 2. A circular cylinder with an open top has a volume of 192π cubic inch. If the cost of the material for the bottom of the cylinder is 15 cents per square inch and the cost of the material for the sides of the cylinder is 5 cents per square inch, what would be the ideal height and radius of a cylinder that would minimize manufacturing costs?

$\underbrace{\text{minimize cost } C}$ Q: $r = ?$ $h = ?$

Surface area!



$SA = \underbrace{(\pi r^2)}_{\text{bottom}} + \underbrace{h(2\pi r)}_{\text{curved surface}}.$

$C = (0.15)(\pi r^2) + (0.05)(2\pi r h) \leftarrow$ 2 vars.

$= (0.15\pi)r^2 + (0.1\pi)\, r \cdot \left(\frac{192}{r^2}\right)$

$C = (0.15\pi)r^2 + (19.2\pi)\left(\frac{1}{r}\right) \leftarrow$ 1 var.

$\frac{dC}{dr} \leftarrow C' = \boxed{(0.15\pi)(2r) + (19.2\pi)\left(-\frac{1}{r^2}\right) = 0}$

$0.3\cancel{\pi}\, r = \frac{19.2\cancel{\pi}}{r^2}$

$r^3 = \frac{19.2}{0.3} = 64$

$r = \sqrt[3]{64} = +4$ inches. $\leftarrow$ CN.

2nd derivative test

Proove CN is minima.

$C'' = 0.3\pi + (19.2\pi)\left(+\frac{2}{r^3}\right) > 0$ $\rightarrow r > 0$

$\therefore$ CN $\rightarrow$ minima.

Ans: For minimum costs.

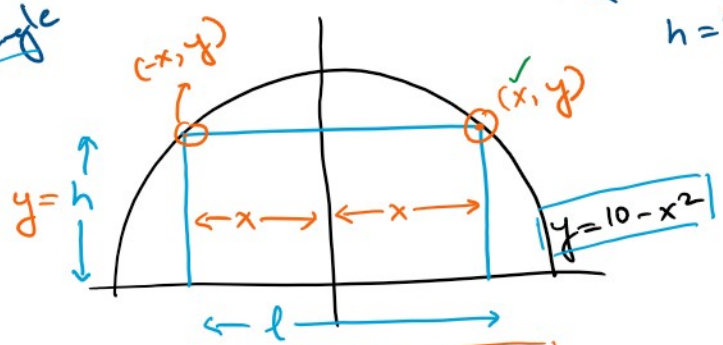
$r = 4$ inches

$h = \frac{192}{r^2} = 12$ inches.

Problem 3. A rectangle has its base on the x−axis and its other two vertices above the x−axis, on the parabola given by $y = 10 - x^2$. What are the dimensions of this rectangle so that it has the largest possible area?

Q: $l = ?$
$h = ?$

maximize area of rectangle



upside down Parabola is $\boxed{\text{Symmetric}}$ about y-axis.

$A = l \cdot h$

$\boxed{l = x + x = 2x}$

$A = (y)(2x) \leftarrow$ 2var.

$A = (10 - x^2)(2x)$

$A = 20x - 2x^3 \leftarrow$ 1 var.

$A' = 20 - 6x^2 = 0$

$x^2 = \frac{20}{6} = \frac{10}{3}$

CN. $x = +\sqrt{10/3}$

Proove CN is optimal:

$A'' = -12x < 0$

CN → maxima.

Ans: For maximum area.

$l = 2x = 2\sqrt{10/3}$

$h = 10 - x^2 = 10 - \frac{10}{3}$

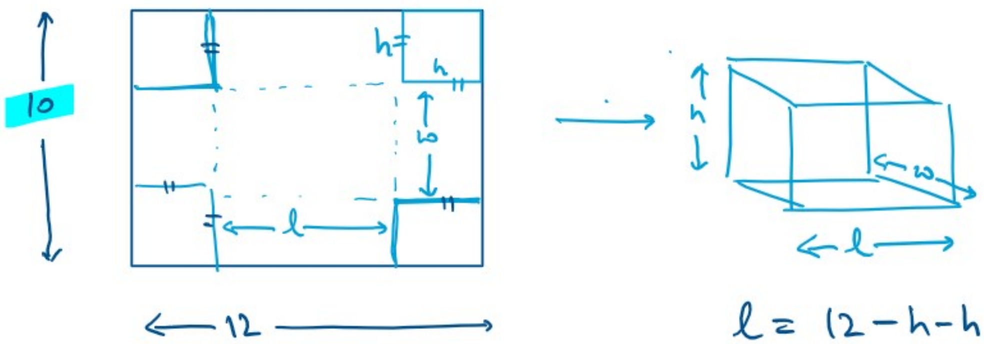
$= \frac{20}{3}$

Problem 4. An open top cardboard box is made from a 10 inch by 12 inch cardboard piece by cutting identical squares from the corners and then by folding up the flaps. Find the dimensions of the box that will maximize its volume.

$V = l \cdot w \cdot h$

Q: $l = ?, w = ?, h = ?$

$V = l \cdot w \cdot h$



$l = 12 - h - h = 12 - 2h$

$w = 10 - h - h = 10 - 2h$

$V = (12-2h)(10-2h) \cdot h \leftarrow$ 1 var only!

$= (120 - 20h - 24h + 4h^2)h$

$V = 120h - 44h^2 + 4h^3$

$V' = \underbrace{120 - 88h + 12h^2} = 0$

$3h^2 - 22h + 30 = 0$

$h = \frac{22 \pm \sqrt{(22)^2 - 4(3)(30)}}{6} = \frac{22 \pm \sqrt{124}}{6}$ → $\cancel{5.5}$ → 1.81 ⇓ maxim.

Proove CN → max.

$V'' = -88 + 24h$

$V''(1.81) < 0$

Ans:
$h = 1.81$ inches.
$l = 12 - 2h = 8.38$ inches
$w = 10 - 2h = 6.38$ inches

Problem 5. Find the most general antiderivative → $g(x)$ for the following:

a) $f(x) = 3x^2(x-4)$

$f(x) = g'(x) = 3x^3 - 12x^2$

$g(x) = (3)\frac{x^{3+1}}{3+1} - (12) \cdot \frac{x^{2+1}}{2+1}$

$= \frac{3x^4}{4} - \frac{\cancel{12}^{4} x^3}{\cancel{3}} + C$

b) $f(x) = (x-1)^2$

$f(x) = g'(x) = (x-1)^2 = x^2 - 2x + 1$

$g(x) = \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + x + C$

c) $f(x) = 5e^x + \frac{\pi}{1+x^2}$

$$f(x) = g'(x) = 5e^x + \pi\left(\frac{1}{1+x^2}\right)$$

$$g(x) = 5e^x + \pi \arctan(x) + C$$

No equivalent product/quotient rule for $\underbrace{\text{antiderivatives}}_{\text{power increases } (+1)}$.

d) $f(x) = \dfrac{1+2x-3x^2}{\sqrt{x}}$

$$f(x) = g'(x) = \frac{1}{\sqrt{x}} + \frac{2x}{\sqrt{x}} - \frac{3x^2}{\sqrt{x}}$$

$$\frac{x^2}{\sqrt{x}} = x^{2-1/2}$$

$$= x^{-1/2} + 2x^{1/2} - 3x^{3/2}$$

$$g(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{2x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 3 \cdot \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}$$

$$= \frac{x^{1/2}}{1/2} + \frac{2x^{3/2}}{3/2} - \frac{3x^{5/2}}{5/2}$$

$$g(x) = F(x) = 2x^{1/2} + \frac{4}{3}x^{3/2} - \frac{6}{5}x^{5/2} + C$$

e) $f(x) = \sqrt[3]{x} + \dfrac{3}{x} + \dfrac{5}{x^2} + \dfrac{7}{\sqrt{1-x^2}} + 9\sin x + 6x^{11}$

$$f(x) = g'(x) = x^{1/3} + \frac{3}{x} + 5x^{-2} + 7\left(\frac{1}{\sqrt{1-x^2}}\right) + 9\sin x + 6x^{11}$$

$$F(x) = g(x) = \frac{x^{1/3+1}}{\frac{1}{3}+1} + 3\ln|x| + \frac{5x^{-2+1}}{-2+1} + 7\arcsin(x)$$

$$+ 9(-\cos x) + 6\frac{x^{11+1}}{11+1}$$

$$F(x) = \frac{3}{4}x^{4/3} + 3\ln|x| - 5x^{-1} + 7\arcsin(x)$$

$$- 9\cos(x) + \frac{\cancel{6}x^{12}}{\cancel{12}_2} + C$$

Problem 6. Find $f(x)$ for the following:

a) $f''(x) = 3x^2 + 6e^x + \dfrac{2}{x^2}$ $= 3x^2 + 6e^x + 2x^{-2}$

Step 1: $f'(x) = \dfrac{\cancel{3}x^3}{\cancel{3}} + 6e^x + \dfrac{2x^{-1}}{(-1)} + C_1$

$f'(x) = x^3 + 6e^x - 2x^{-1} + C_1$

Ans.

Step 2: $\boxed{f(x) = \dfrac{x^4}{4} + 6e^x - 2\ln|x| + C_1 x + C_2}$

b) $f'(x) = \dfrac{6}{x} + e^x - 4$, $f(1) = 6$

$f(x) = 6\ln|x| + e^x - 4x + C_1$

$f(1) = 6\cancel{\ln|1|}^{0} + e^1 - 4(1) + C_1 = 6$

$e - 4 + C_1 = 6$

$C_1 = 10 - e$

Ans: $f(x) = 6\ln|x| + e^x - 4x + 10 - e$

2 initial conditions

c) $f''(x) = \sin x + \cos x$, $f(0) = 3$, $f'(0) = 4$

step 1. $f'(x) = -\cos(x) + \sin(x) + C_1$

$4 = f'(0) = -\cancel{\cos}(0)^{1} + \cancel{\sin}(0)^{0} + C_1$

$C_1 = 4 + 1 = 5$

$f'(x) = -\cos(x) + \sin(x) + 5$

step 2 $f(x) = -\sin(x) - \cos(x) + 5x + C_2$

$3 = f(0) = -\cancel{\sin}(0)^{0} - \cancel{\cos}(0)^{1} + \cancel{5}0^{0} + C_2$

$C_2 = 3 + 1 = 4$

Ans: $\boxed{f(x) = -\sin(x) - \cos(x) + 5x + 4}$

d) $f''(x) = 24x^2 + 6x + 4$, $f(0) = 3$, $f(1) = 10$

$f'(x) = \frac{24 \cdot x^3}{3} + \frac{6x^2}{2} + 4x + C_1 = 8x^3 + 3x^2 + 4x + C_1$

$f(x) = \frac{8x^4}{4} + \frac{3x^3}{3} + \frac{4x^2}{2} + C_1 x + C_2$

$f(x) = 2x^4 + x^3 + 2x^2 + \cancel{C_1}^{2}x + \cancel{C_2}^{3}$

$3 = f(0) = 0 + 0 + 0 + 0 + C_2 \Rightarrow C_2 = 3$

$10 = f(1) = 2 + 1 + 2 + C_1 + 3$

$10 = 8 + C_1 \Rightarrow C_1 = 2$

Ans: $\boxed{f(x) = 2x^4 + x^3 + 2x^2 + 2x + 3}$

Problem 7. If the acceleration of a particle is given by $\vec{a}(t) = <\cos t, t>$, $\vec{v}(0) = <2, 3>$ and $\vec{r}(0) = <1, 1>$, find the position function of the particle at any given time t.

$\bar{r}(t) \to f(x)$
$\bar{v}(t) \to f'(x)$
$\bar{a}(t) \to f''(x)$

$f''(x) \Rightarrow \bar{a}(t) = \langle \cos t, t \rangle$

$f'(x) \Rightarrow \bar{v}(t) = \langle \sin t + C_1, \frac{t^2}{2} + C_2 \rangle$

$\bar{v}(0) = \langle \cancel{\sin(0)} + C_1 + 0 + C_2 \rangle$

$= \langle C_1, C_2 \rangle = \langle 2, 3 \rangle.$

$\therefore C_1 = 2, \ C_2 = 3$

$\bar{v}(t) = \langle \sin t + 2, \frac{t^2}{2} + 3 \rangle$

$f(x) \Rightarrow \bar{r}(t) = \langle -\cos t + 2t + C_3, \frac{1}{2} \cdot \frac{t^3}{3} + 3t + C_4 \rangle$

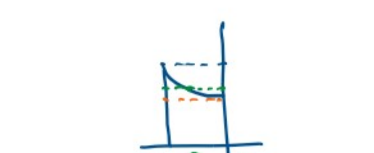
$\bar{r}(0) = \langle -\cancel{\cos(0)} + 0 + C_3, 0 + 0 + C_4 \rangle$

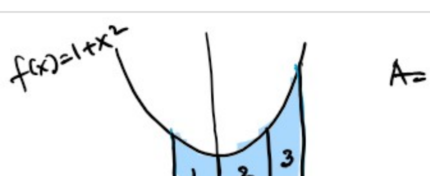
$= \langle 1, 1 \rangle.$

$-1 + C_3 = 1 \qquad C_4 = 1$

$C_3 = 2$

Ans: $\boxed{\bar{r}(t) = \langle -\cos t + 2t + 2, \frac{t^3}{6} + 3t + 1 \rangle}$





$A = (\Delta x) \cdot h$

Problem 8. Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using
a) three rectangles and right endpoints.

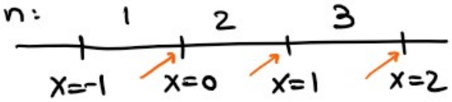
$[a, b] = [-1, 2].$

$n = 3$

$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{3} = \frac{3}{3} = 1$

$\Delta x = \frac{b-a}{n} = \frac{\quad}{3} = \frac{3}{3} = 1$

$\boxed{f(x) = 1 + x^2}$



(right endpoints)

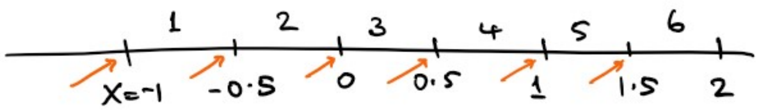
n	x_i^*	$f(x_i^*)$	$A = f(x_i^*)\,\Delta x$
1	$x=0$	$f(0) = 1$	$1(1) = 1$
2	$x=1$	$f(1) = 2$	$2(1) = 2$
3	$x=2$	$f(2) = 5$	$5(1) = 5$

Total Area $= 1 + 2 + 5$
$= 8$ sq. units

b) six rectangles and left endpoints.

$n = 6$

$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{6} = \frac{3}{6} = \frac{1}{2} = 0.5$



n	x_i^*	$f(x_i^*)$	$A = f(x_i^*)(\frac{1}{2})$
1	(-1)	$f(-1) = 2$	$2(\frac{1}{2}) = 1$
2	(-0.5)	$f(-0.5) = 5/4$	$\frac{5}{4}(\frac{1}{2}) = 5/8$
3	0	$f(0) = 1$	$1/2$
4	0.5	$f(0.5) = 5/4$	$\longrightarrow$ 5/8
5	1	$f(1) = 2$	1
6	1.5	$f(1.5) = \frac{13}{4}$	$(\frac{13}{4})(\frac{1}{2}) = \frac{13}{8}$

Total area $=$
$\frac{5}{8} + \frac{5}{8} + \frac{13}{8} + \frac{1}{2} + 2$
$= \frac{43}{8}$ Ans.
~ 5

Problem 9. Approximate the area under the curve $f(x) = e^{x^2}$ in the interval $[0, 1]$ using $n = 4$ and the midpoint rule.

Problem 10. Express the area under the curve $f(x) = \dfrac{2x}{x^2+1}$ on the interval $[1, 3]$ as a limit, using equally spaced n partitions and right endpoints.