Exam 3 Review (Sections 4.1 - 5.2)

CN: $f'(x) = 0$ / $f'(x)$ DNE

$\frac{1}{7} - 1 = -\frac{6}{7}$

Problem 1. Find the critical numbers for the function $f(x) = (x-2)^{\frac{1}{7}}x^2$.

$$f'(x) = (x-2)^{1/7}(2x) + (x^2)\frac{1}{7}(x-2)^{-6/7} = 0$$

$$= (x-2)^{1/7}(2x) + \frac{x^2}{7(x-2)^{6/7}}$$

$$= \frac{7(x-2)^{6/7}(x-2)^{1/7}(2x) + x^2}{7(x-2)^{6/7}}$$

$(x-2)(2x)(7) = 14x^2 - 28x$

$14x^2 - 28x + x^2 = 15x^2 - 28x = x(15x-28)$

$$f'(x) = \frac{14x^2 - 28x + x^2}{7(x-2)^{6/7}} = \frac{x(15x-28)}{7(x-2)^{6/7}} = 0$$

$x = 0$, $x = \frac{28}{15}$, $x = 2$

$f'(x) = 0$ when $x = 0, \frac{28}{15}$; $f'(x)$ DNE @ $x = 2$

CNs are $x = 0, \frac{28}{15}, 2$

Problem 2. Find the inflection points for the function $f(x) = x^5 + 10x^4$.

IPs: → $f''(x) = 0$
→ $f''(x)$ DNE
+ and changes sign around the point

$$f(x) = x^5 + 10x^4$$
$$f'(x) = 5x^4 + 40x^3$$
$$f''(x) = 20x^3 + 120x^2 = 0$$
$$f''(x) = 20x^2(x+6) = 0$$

$x = 0$ $x = -6$

$f''(x)$: (+)(−) (+)(+) (+)(+)

−10 −6 −1 0 2

− + +

IP ~~IP~~

Inflection pt @ $x = -6$

Problem 3. Given the function $f(x) = \dfrac{\sqrt{x-4}}{x+1}$, find the following:

$-1 \quad 4$

a) Domain

$x \neq -1$; $(x-4) \geq 0 \Rightarrow x \geq 4$; $\boxed{D: [4, \infty)}$

b) Vertical asymptotes

$\boxed{\text{none}}$ (since $x = -1$ is outside the D)

c) Horizontal asymptotes $\quad \lim_{x\to\infty} f(x)$ and $\cancel{\lim_{x\to(-\infty)} f(x)}$

$$\lim_{x\to\infty} \frac{\sqrt{x-4}}{x+1} \sim \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = \frac{1}{\infty} = 0$$

$\boxed{\text{HA is } f(x) \to 0 \text{ as } x \to \infty}$

d) $x-$intercept(s) and $y-$intercept.

set $y=0$
$\sqrt{x-4} = 0$
$x = 4$ $\Big\}$ $(4, 0)$

set $x=0$
But 0 is outside D
$\therefore$ no y-intercepts.

e) Critical numbers and intervals where the function is increasing and decreasing.

$$f'(x) = \frac{(x+1)\frac{1}{2\sqrt{x-4}} - \sqrt{x-4}\,(1)}{(x+1)^2} = \frac{(x+1) - 2\sqrt{x-4}\,\sqrt{x-4}}{2\sqrt{x-4}\,(x+1)^2}$$

$(x+1) - 2(x-4)$
$= x+1-2x+8$
$= 9-x$

$$f'(x) = \frac{9-x}{2\sqrt{x-4}\,(x+1)^2} = 0$$

$x=9$

$x=4$ $\quad$ $\cancel{x=-1}$

$f'(x)$ $\quad$ $(+)(+)(+)$ $\quad$ $(-)(+)(+)$

$4 \quad 5 \quad 9 \quad 10$

$(+) \quad (-)$

CN: $x = 9 \to$ local maxima.

~~f) Inflection points and intervals where the function is concave up and concave down.~~

$f(x)$ increasing in $[4, 9)$

$f(x)$ decreasing in $(9, \infty)$

Problem 4. Find the absolute maxima and the absolute minima for the following:

a) $f(x) = (5 + \ln x)^4$, $[\frac{1}{e^7}, 1]$. $\rightarrow [e^{-7}, 1]$

$$f'(x) = 4(5+\ln x)^3 \cdot \frac{1}{x} = \frac{4(5+\ln x)^3}{x} = 0$$

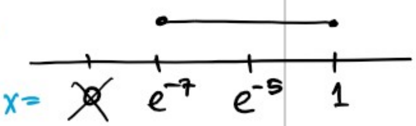
$\nearrow x = e^{-5}$; $\searrow x = 0$

$5 + \ln x = 0$

$\ln x = -5$

$x = e^{-5} = \frac{1}{e^5}$

CN: $x = e^{-5}$



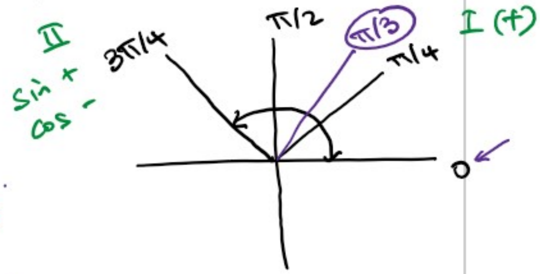
$$\begin{cases} f(e^{-7}) = [5 + \ln(e^{-7})]^4 = (5-7)^4 = +16 \\ f(1) = [5 + \ln(1)]^4 = 625 \rightarrow \text{max} \end{cases}$$

CN: $f(e^{-5}) = [5 + \ln(e^{-5})]^4 = (5-5)^4 = 0 \leftarrow \text{min}$

$\ln e^{-7} = -7\ln e = -7$

Absolute maxima: 625 $\leftarrow y$; Absolute minima: 0 $\leftarrow y$.

b) $f(x) = \sin^2 x + \cos x$, $[0, \frac{3\pi}{4}]$.



$$f'(x) = 2\sin x \cos x - \sin x = 0$$

$$= \sin(x)[2\cos x - 1] = 0$$

$x = 0$; $\cos x = \frac{1}{2}$, $x = \pi/3$

$$\begin{cases} f(0) = 0 + 1 = 1 \\ f(\frac{3\pi}{4}) = [\sin(\frac{3\pi}{4})]^2 + \cos(\frac{3\pi}{4}) = (\frac{\sqrt{2}}{2})^2 - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} < 1 \rightarrow \text{min} \end{cases}$$

CN: $f(\frac{\pi}{3}) = [\sin(\frac{\pi}{3})]^2 + \cos(\frac{\pi}{3}) = (\frac{\sqrt{3}}{2})^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4} \rightarrow \text{max.}$

Absolute max: $\frac{5}{4}$ Absolute min: $\frac{\sqrt{2}}{2}$

Problem 5. The graph of the first derivative $f'(x)$ of a function $f(x)$ is shown.

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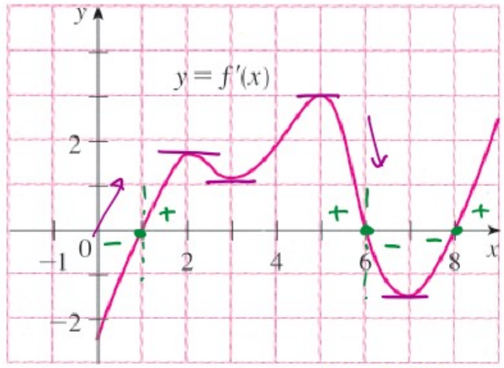
$f \to f' \to f''$

$g \to g'$

(+)ve

(-)ve



$f'(x) > 0$ $\quad$ $f'(x) < 0$

a) On what intervals is f increasing? Decreasing?

f increasing: $(1, 6) \cup (8, \infty)$

f decreasing: $(0, 1) \cup (6, 8)$

CNs: $x = 1, 6, 8$

b) Where does f have a local maxima? A local minima?

$x = 6$ $\quad$ $x = 1$, $x = 8$

− + min

max + −

c) On what intervals is f concave up? Concave down?

$f''(x) > 0 \to f'(x)$ is increasing

f concave up: $(0, 2) \cup (3, 5) \cup (7, \infty)$

$f''(x) < 0$ or f' decreasing

$(2, 3) \cup (5, 7)$

d) What are the inflection points of f?

$f'(x) = 0$

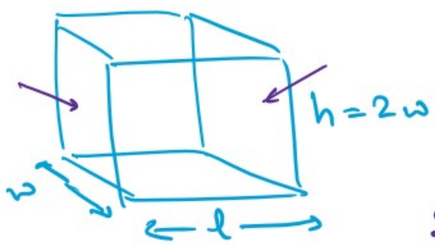
@ $x = 2, 3, 5, 7$

only 5 sides

$\boxed{h=2w}$

Problem 6. A box with an open top has a volume of 400 cubic meters. If the height of the box is twice its width, find the dimensions of a box which would have the smallest possible surface area.

→ minimize.



$$\left.\begin{aligned} V &= l\cdot w\cdot h \\ &= l\cdot w(2w) \end{aligned}\right\} \text{constraint}$$

$$\boxed{400 = 2lw^2} \leftarrow \quad l = \frac{400}{2w^2} = \frac{200}{w^2}$$

$$\begin{aligned} SA &= lw + 2wh + 2lh \\ &= lw + 2w(2w) + 2l(2w) \\ &= 5lw + 4w^2 \\ &= 5\left(\frac{200}{w^2}\right)w + 4w^2 \end{aligned}$$

$$SA = \frac{1000}{w} + 4w^2$$

CN: $\boxed{SA' = -\frac{1000}{w^2} + 8w} = 0$

$$8w = \frac{1000}{w^2}$$

$$w^3 = \frac{1000}{8} = 125$$

$$\boxed{\begin{aligned} w &= \sqrt[3]{125} = 5m \\ h &= 2w = 10m \\ l &= \frac{200}{25} = 8m \end{aligned}}$$

2nd derivative test.

Check CN → minima.

$SA'' = (-1000)\left(-\frac{2}{w^3}\right) + 8$

$= \frac{2000}{w^3} + 8 > 0$

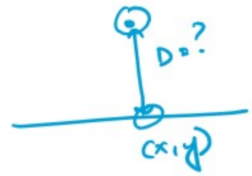
w (length) must be > 0.

$\therefore$ w → local minima!

$SA''(w=5) = \left(\frac{2000}{125} + 8\right) > 0$.

(x, y)

Problem 7. Find the point on the parabola $y = \frac{1}{2}x^2$ that is closest to the point $(5, 1)$.



$$D^2 = (y - y_1)^2 + (x - x_1)^2$$

$$D^2 = (y-1)^2 + (x-5)^2$$

$$D^2 = \left(\frac{x^2}{2} - 1\right)^2 + (x-5)^2$$

CN: $\cancel{2}D \cdot \underbrace{\frac{dD}{dx}}_{=0} = \underbrace{\cancel{2}\left(\frac{x^2}{2} - 1\right)\left(\frac{1}{\cancel{2}}\cancel{2}x\right) + \cancel{2}(x-5)(1)}_{} = 0$

$$\frac{x^3}{2} \underbrace{- x + x}_{} - 5 = 0$$

$$\frac{x^3}{2} - 5 = 0$$

$$x^3 = 2(5) = 10$$

$$x = \sqrt[3]{10}$$

$$y = \frac{1}{2}x^2 = \frac{1}{2}\left(\sqrt[3]{10}\right)^2$$

Ans: Point $\left(\sqrt[3]{10}, \frac{1}{2}\left(\sqrt[3]{10}\right)^2\right)$

$L'H \to \frac{0}{0}$, $\to \frac{\infty}{\infty}$

$\csc^2(x) = \frac{1}{\sin^2(x)}$

Problem 8. Find the following limits:

$1 = y =$ a) $\lim_{x\to 0^+} (\sin x)^{\tan x} \xrightarrow{x=0^+} (\sin 0^+)^{\tan(0^+)} \sim 0^0 \to$ like log diff.

$\ln y = \lim_{x\to 0^+} \left[\underbrace{\tan(x)}\cdot \underbrace{\ln(\sin x)}\right] \sim (0)(-\infty)$

$= \lim_{x\to 0^+}\left[\frac{\ln(\sin x)}{\left(\frac{1}{\tan(x)}\right)}\right] = \lim_{x\to 0^+}\left[\frac{\ln(\sin x)}{\cot x}\right] \sim \frac{-\infty}{\infty}$ ✓

$\xrightarrow{L'H} \frac{\frac{1}{\sin x}\cdot\cos(x)}{-\csc^2(x)} = -\frac{\cos(x)}{\sin(x)}\cdot \sin^2(x) = -\cos(x)\sin(x)$

$\lim_{x\to 0^+} -\cos(x)\sin(x) = -(1)(0) = 0$

$\ln y = 0$

$\therefore y = e^0 = 1.$ Ans.

$\frac{y}{5} =$ b) $\lim_{x\to\infty} \cancel{5}\left(1+\frac{3}{x}\right)^{2x} \xrightarrow{x\to\infty} \left(1+\frac{\cancel{3}}{\infty}\right)^{\infty} \sim 1^\infty$

$\ln\left(\frac{y}{5}\right) = \lim_{x\to\infty} \underbrace{(2x)}\cdot \underbrace{\ln\left(1+\frac{3}{x}\right)} \sim (\infty)\ln(1)^0 \sim (\infty)(0)$

$= \lim_{x\to\infty}\left[\frac{\ln\left(1+\frac{3}{x}\right)}{\left(\frac{1}{2x}\right)}\right] \sim \frac{0}{0}$ ✓

$\frac{1}{2}\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\left(\frac{1}{2}\right)$

$\frac{d}{dx}\left(1+\frac{3}{x}\right) = 0 + 3\left(\frac{-1}{x^2}\right)$

$\xrightarrow{L'H} \frac{\left(\frac{1}{1+\frac{3}{x}}\right)\cdot\left(-\frac{3}{x^2}\right)}{\left(\frac{-1}{2x^2}\right)} = \frac{1}{\left(1+\frac{3}{x}\right)}\cdot\left(\frac{\cancel{-}3}{\cancel{x^2}}\right)\left(\frac{\cancel{-}2\cancel{x^2}}{1}\right)$

$\lim_{x\to\infty} \frac{+6}{\left(1+\frac{3}{x}\right)^0} = 6 = \ln\left(\frac{y}{5}\right)$ Ans.

$\frac{y}{5} = e^6 \quad \therefore \boxed{y = 5e^6}$

c) $\lim_{x\to 1^+} \underbrace{(x-1)}\underbrace{\tan\left(\frac{\pi x}{2}\right)} \xrightarrow{x=1^+} (1-1)\cdot\tan\left(\frac{\pi}{2}\right) \sim (0)(\infty)$

c) $\lim_{x\to 1^+}(x-1)\tan\left(\frac{\pi x}{2}\right) \xrightarrow{x=1^+} (1-1)\cdot\tan\left(\frac{\pi}{2}\right) \sim (0)(\infty)$

← move

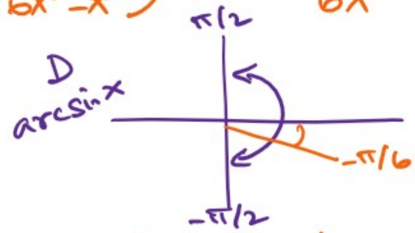
$$= \lim_{x\to 1^+}\left[\frac{(x-1)}{\frac{1}{\tan\left(\frac{\pi x}{2}\right)}}\right] = \lim_{x\to 1^+}\frac{(x-1)}{\cot\left(\frac{\pi x}{2}\right)} \sim \frac{0}{0}$$

$$\xrightarrow{L'H} \frac{1}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\left(\frac{\pi}{2}\right)} = \lim_{x\to 1^+}\left(-\frac{2}{\pi}\right)\cancelto{1}{\sin^2\left(\frac{\pi x}{2}\right)}$$

$$= -\frac{2}{\pi}$$ Ans.

d) $\lim_{x\to\infty}\arcsin\left(\frac{1-3x^2}{6x^2-x}\right)$

$$\lim_{x\to\infty}\left(\frac{1-3x^2}{6x^2-x}\right) \sim \frac{-3x^2}{6x^2} = -\frac{1}{2}$$



$\sin\theta = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}$

↓

$\arcsin\left(-\frac{1}{2}\right)$

↓

$\arcsin\left(\sin\left(-\frac{\pi}{6}\right)\right)$

$= -\frac{\pi}{6}$ Ans.

$(3,0)$ $\quad x=3 \rightarrow CN$ of $f(x)$

Problem 9. Given $f(3)=0$, $f'(3)=0$, $\underbrace{f''(3)=-4}_{f''(CN)<0}$ what can you say about $f(x)$ at $x=3$?

$\therefore CN \rightarrow$ local maxima.

$f(x)$ has a local maxima @ $(3,0)$

CN @ $x=3$

Problem 10. Find the value of c that satisfies the Mean value Theorem for $f(x)=\ln(x^3)$, $[1,4]$. $\rightarrow [a,b]$

$$f'(x) = \frac{1}{x^3}\cdot(3x^2) = \frac{3}{x}$$

$$f'(c) = \frac{3}{c} = \frac{f(b)-f(a)}{b-a} = \frac{\ln(4^3) - \cancel{\ln(1^3)}^{0}}{(4-1)}$$

$$\frac{3}{c} = \frac{\cancel{3}\ln 4}{\cancel{3}} \Rightarrow 1 < \left(c = \frac{3}{\ln(4)}\right) < 4$$

Problem 11. Find the most general antiderivative for $f(x) = 15x^4 + \sqrt[3]{x^2} + \frac{\pi}{x} + \frac{2}{x^2} + \frac{5}{1+x^2} + 9\sec^2 x$.

$\sqrt[3]{x^2} = x^{2/3}$

$\frac{1}{x^2} = x^{-2}$

$$F(x) = \frac{\cancel{15}^{3}x^5}{\cancel{5}} + \frac{x^{2/3+1}}{(2/3+1)} + \pi\ln|x| + 2\cdot\frac{x^{-2+1}}{(-2+1)}$$

$$+ 5\arctan(x) + 9\tan(x)$$

$$= 3x^5 + \frac{3}{5}x^{5/3} + \pi\ln|x| - \frac{2}{x} + 5\arctan(x)$$

$$+ 9\tan(x) + C$$

$f'' \longrightarrow f' + c_1 \longrightarrow f + c_1 x + c_2$

Problem 12. Find $f(x)$ if $f''(x) = 4x^3 + 2\cos x$, $f(0) = 0$, $f'(0) = 4$.

$f'(x) = \cancel{4} \cdot \frac{x^4}{\cancel{4}} + 2\sin(x) + c_1$

$4 = f'(0) = 0 + 0 + c_1 \Rightarrow c_1 = 4$

$f'(x) = x^4 + 2\sin(x) + 4$

$f(x) = \frac{x^5}{5} - 2\cos(x) + 4x + c_2$

$0 = f(0) = 0 - 2(1) + 0 + c_2 \Rightarrow c_2 = 2$

$f(x) = \frac{1}{5}x^5 - 2\cos(x) + 4x + 2$

Problem 13. Use geometry to evaluate $\int_{-1}^{2} (1 - x)\, dx$. $= \int_{x=-1}^{x=2} f(x)\, dx$

$f(x) = 1 - x$

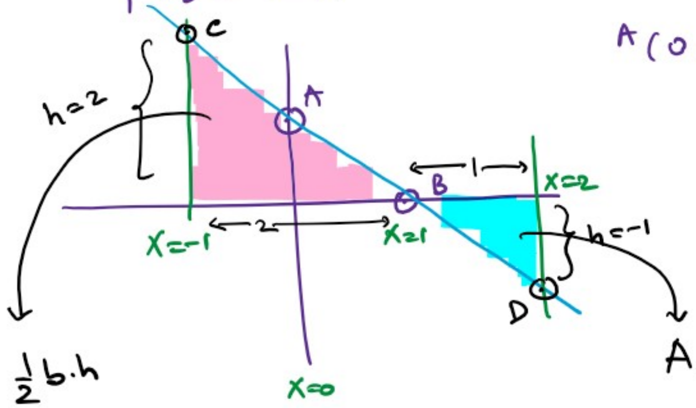
$D = (2, -1)$

$A(0,1)$, $B(1,0)$

$C(-1,2)$

$LB \Rightarrow x = -1$

$UB \Rightarrow x = 2$



$A_1 = \frac{1}{2} b \cdot h$

$= \frac{1}{2}(2)(2)$

$= +2$

$A_2 = \frac{1}{2} bh$

$= \frac{1}{2}(1)(-1)$

$= -\frac{1}{2}$

$\int_{-1}^{2} (1-x)\,dx = (2) + \left(-\frac{1}{2}\right)$

$= +3/2$

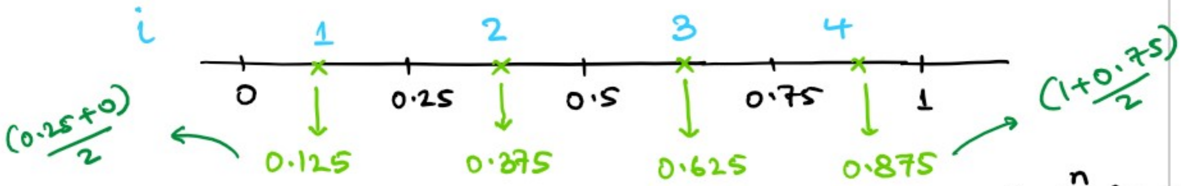
Problem 14. Approximate the area under the curve $f(x) = e^{x^2}$ in the interval $[0, 1]$ using $n = 4$ and the midpoint rule.

Reimann sum

Problem 14. Approximate the area under the curve $f(x) = e^{x^2}$ in the interval $[0, 1]$ using $n = 4$ and the midpoint rule.

$f(x) = e^{x^2}$

width $\leftarrow \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$



$A = \sum_{i=1}^{n} f(x_i^*) \Delta x$

$\Delta x = 0.25$

i	x_i^*	$f(x_i^*) = ht$
1	0.125	$e^{(0.125)^2}$
2	0.375	$e^{(0.375)^2}$
3	0.625	$e^{(0.625)^2}$
4	0.875	$e^{(0.875)^2}$

$$A = (0.25)\left[e^{(0.125)^2} + e^{(0.375)^2} + e^{(0.625)^2} + e^{(0.875)^2}\right]$$

~ 1.4487 square units

Problem 15. Express the exact area under the curve $f(x) = \frac{2x}{x^2+1}$ on the interval $[1, 3]$ as a limit.

x_i^*

$a=1$

$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

$x_i^* = a + i\Delta x$
$= 1 + \frac{2i}{n}$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x) f(x_i^*)$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\right)\left[\frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}\right]$$

UB-LB

no longer require c

Problem 16. Evaluate the following definite integrals.

a) $\int_1^4 \frac{x^2-1}{x\sqrt{x}}\,dx$

$f'(x) = \frac{x^2-1}{x\sqrt{x}} = \frac{x^2}{x\sqrt{x}} - \underbrace{\frac{1}{x\sqrt{x}}}_{x^{3/2}} = x^{1/2} - x^{-3/2}$

$f(x) = \frac{x^{1/2+1}}{\left(\frac{1}{2}+1\right)} - \frac{x^{-3/2+1}}{(-3/2+1)}$

$= \frac{2}{3}x^{3/2} + 2x^{-1/2} + \cancel{c}$

$= \left[\frac{2}{3}x^{3/2} + \frac{2}{\sqrt{x}}\right]\Big|_{x=4} - \left[\frac{2}{3}x^{3/2} + \frac{2}{\sqrt{x}}\right]\Big|_{x=1}$

$\left(\frac{2}{3}(4)^{3/2} + \frac{2}{\sqrt{4}}\right) - \left(\frac{2}{3}(1)^{3/2} + \frac{2}{\sqrt{1}}\right)$

$= \frac{2}{3}\cdot 8 + \frac{\cancel{2}^{1}}{\cancel{2}} - \frac{2}{3} - 2$

$= \frac{14}{3} - 1 = \frac{11}{3}$ Ans.

b) $\int_1^2 \left(\frac{3}{x} - 2^x\right) dx$

$= 3\ln|x| + \frac{2^x}{\ln(2)}\Big|_{x=1}^{x=2}$

$= \left(3\ln 2 + \frac{\cancel{2^2}^{4}}{\ln(2)}\right) - \left(3\cancel{\ln 1}^{0} + \frac{2^1}{\ln(2)}\right)$

$A = 3\ln 2 + \frac{4}{\ln(2)} - \frac{2}{\ln(2)} = 3\ln(2) + \frac{2}{\ln(2)}$ Ans.