WIR 5

Week-in-Review 5: Exam 1 review

Problem 1. Find the angle between the vectors $< 3, 1 >$ and $-2\vec{i} + 2\vec{j}$.

$\bar{a} = \langle 3, 1 \rangle$ $\qquad$ $\bar{b} = \langle -2, 2 \rangle$

dot product : $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos\theta$

$= a_1 b_1 + a_2 b_2$

$$\cos\theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} = \frac{\langle 3, 1 \rangle \langle -2, 2 \rangle}{|\langle 3, 1 \rangle| \, |\langle -2, 2 \rangle|}$$

$$\theta = \arccos\left(\frac{-6+2}{\sqrt{10}\sqrt{8}}\right) = \arccos\left(\frac{-4}{\sqrt{80}}\right)$$

Problem 2. What value(s) of x will make the vectors $x\vec{i} + \vec{j}$ and $(4 + x)\vec{i} + 3\vec{j}$ orthogonal?

$\vec{a} = \langle x, 1 \rangle$ $\qquad$ $\bar{b} = \langle (4+x), 3 \rangle$

$\bar{a} \cdot \bar{b} = 0$

$\theta = 90^\circ$

$\bar{a} \cdot \bar{b} = 0$

$\langle x, 1 \rangle \langle (4+x), 3 \rangle = 0$

$x(4+x) + 1(3) = 0$

$4x + x^2 + 3 = 0$

$x^2 + 4x + 3 = 0$

$(x+3)(x+1) = 0$

$x + 3 = 0$ ie $x = -3$

$x + 1 = 0$ ie $x = -1$

Ans.

$\vec{b}$ $\bar{a} =$

Problem 3. Find a vector of length 5 in the direction of the vector $< -3, 2 >$.

$$\hat{a} = \text{unit vector } a = \frac{\bar{a}}{|\bar{a}|} = \underbrace{\frac{\langle -3, 2\rangle}{\sqrt{13}}}_{\text{direction of } \vec{a}}$$

$$\vec{b} = 5(\hat{a}) = 5\left(\frac{\bar{a}}{|\bar{a}|}\right)$$

$$\vec{b} = 5\left(\frac{\langle -3, 2\rangle}{\sqrt{13}}\right) = \frac{5}{\sqrt{13}}\langle -3, 2\rangle = \left\langle \frac{-15}{\sqrt{13}}, \frac{10}{\sqrt{13}} \right\rangle$$

A

Problem 4. Find the parametric equations of a line passing through the point $(1, 3)$ and perpendicular to the line $y = -3x + 1$.

$\bar{r}_0 = \langle 1, 3\rangle$

$m_1 = (-3)$

$\bar{v} = \langle x, y\rangle$

$m = \frac{y}{x}$

$m_1 m_2 = -1$

$m_2 = \frac{-1}{m_1} = \frac{-1}{-3} = \frac{1}{3}$ ← slope of line you want.

$\bar{r} = \bar{r}_0 + t\bar{v}$

$\langle 3, 1\rangle$

$m = \frac{y\text{ comp}}{x\text{ comp}} = \frac{\text{rise}}{\text{run}}$

$$\bar{r} = \langle 1, 3\rangle + t\langle 3, 1\rangle$$
$$= \langle 1, 3\rangle + \langle 3t, t\rangle$$
$$= \langle (1+3t), (3+t)\rangle$$

$$\boxed{\begin{aligned} x(t) &= 1 + 3t \\ y(t) &= 3 + t \end{aligned}}$$

Parametric Eq$^{\underline{n}}$.

Problem 5. How much work is done by a force of 10 N in order to push a box 15 m up a ramp.

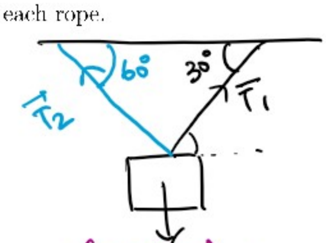
Problem 5. How much work is done by a force of 10 N in order to push a box 15 m up a ramp, given that the ramp is inclined at an angle of 45° to the horizon?

$$\vec{W} = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta$$
$$= (10N)(15m)\cos(45^\circ)$$
$$= 10 \cdot 15 \cdot \frac{\sqrt{2}}{2}$$
$$= 75\sqrt{2} \text{ N-m (or J)}$$

$F = ma$

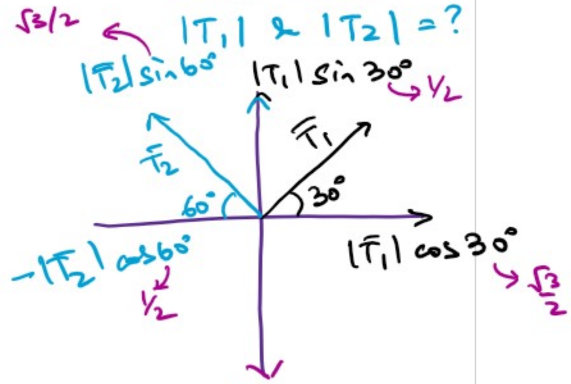
Problem 6. Two ropes are used to suspend a 100 Kg weight. One rope makes an angle of 30° with the horizon while the other makes an angle of 60° with the horizon. Find the magnitude of tension in each rope.

$|T_1|$ & $|T_2| = ?$



$(100kg)\,g$

$g = 9.8\, m/s^2$



System in equilibrium

$$x: \quad |\vec{T}_1| \frac{\sqrt{3}}{2} = |\vec{T}_2| \frac{1}{2} \Rightarrow \boxed{|\vec{T}_2|} = |\vec{T}_1| \frac{(\sqrt{3}/2)}{(1/2)} = \boxed{\sqrt{3}\,|\vec{T}_1|}$$

$100\,g$

$$y: \quad |\vec{T}_1|\left(\frac{1}{2}\right) + |\vec{T}_2|\left(\frac{\sqrt{3}}{2}\right) = 100g$$
$$|\vec{T}_1|\frac{1}{2} + \sqrt{3}\,|\vec{T}_1|\left(\frac{\sqrt{3}}{2}\right) = 100g$$
$$|\vec{T}_1|\frac{1}{2} + |\vec{T}_1|\left(\frac{3}{2}\right) = 100g$$
$$2|\vec{T}_1| = 100g$$

Ans.

$$\boxed{\begin{aligned} |\vec{T}_1| &= 50g \\ |\vec{T}_2| &= \sqrt{3}\,|T_1| \\ &= \sqrt{3}\,50g \end{aligned}}$$

$\bar{r} = \bar{r}_0 + t\bar{v}$

A $\xrightarrow{\bar{v}}$ B

Problem 7. Find the vector equation of a line that passes through the points (2, 5) and (4, 7).

vector eqn of a line is not unique!

$$\bar{v} = \langle B - A \rangle = \langle (4,7) - (2,5) \rangle = \langle 2, 2 \rangle$$
$$\boxed{\bar{v} = \langle 2, 2 \rangle} \rightarrow m = \frac{2}{2} = 1$$
$$\bar{r}_0 = \langle 2, 5 \rangle$$

a line unique

$\bar{r}_0 = \langle 2,5 \rangle$

$$\boxed{\bar{r} = \langle 2,5 \rangle + t\langle 2,2 \rangle} \quad \text{OR} \quad \bar{r} = \langle 4,7 \rangle + t\langle 2,2 \rangle$$

$$= \langle (2+2t),(5+2t) \rangle \quad \text{OR} \quad \bar{r} = \langle 4,7 \rangle + t\langle -2,-2 \rangle$$

$$\text{OR} \quad \bar{r} = \langle 2,5 \rangle + t\langle -2,-2 \rangle$$

$$\boxed{\begin{aligned} x(t) &= 2+2t \\ y(t) &= 5+2t \end{aligned}} \longrightarrow \quad y-5 = 1(x-2)$$

$$\boxed{y = x+3}$$

Problem 8. Express $\tan(\arcsin(2x))$ in terms of x.

Let, $\boxed{\arcsin(2x) = \theta}$

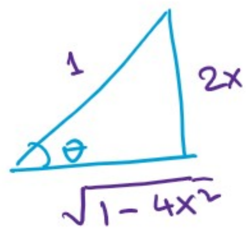
then Q: what is $\tan(\theta)$?

$\sin^{-1}(2x) = \theta$

$\cancel{\sin}(\cancel{\sin^{-1}}(2x)) = \sin\theta$

$2x = \sin\theta$

$\frac{2x}{1} \sim 2x = \sin\theta \sim \frac{O}{H}$



$$\tan\theta = \frac{O}{A} = \frac{2x}{\sqrt{1-4x^2}} \quad \text{Ans.}$$

$$\boxed{O^2 + A^2 = H^2}$$

Problem 9. Find the vertical and horizontal asymptotes of the function $f(x)$. Where is $f(x)$ discontinuous? When is the discontinuity removable?

VA HA

a $f(x) = \frac{x^2+6x+5}{x^2-3x-4}$. $= \frac{(x+5)\cancel{(x+1)}}{(x-4)\cancel{(x+1)}}$

D: $x \neq -1$, $x \neq 4$

$(-\infty,-1)\cup(-1,4)\cup(4,\infty)$

$\boxed{x=4}$ VA non removable.

$\boxed{x=-1 \to \text{hole}}$ removable discontinuity

HA:

$\lim_{x\to\infty} f(x) \sim \frac{x^2}{x^2} = 1$

$\lim_{x\to(-\infty)} f(x) \sim \frac{x^2}{x^2} = 1$

$f(x)$ has HA @ $y=1$ as $x \to \pm\infty$

b $f(x) = \frac{\sqrt{x^2+2}}{3x-6}$

$f(x) = \frac{\sqrt{x^2+2}}{3(x-2)}$

has a VA @ $x=2$ (no holes)

Ans.

HA:

$\lim_{x\to+\infty} f(x) \simeq \frac{\sqrt{x^2}}{3x} = \frac{|x|}{3x} = \frac{+x}{3x} = \frac{1}{3}$

HA $y=1/3$ as $x\to\infty$

Ans.

$\lim_{x\to(-\infty)} f(x) \simeq \frac{\sqrt{x^2}}{3x} = \frac{|x|}{3x} = \frac{-\cancel{x}}{3\cancel{x}} = -\frac{1}{3}$

Ans.

$\boxed{\text{HA @ } y=-1/3 \text{ as } x\to(-\infty)}$

$\boxed{\sqrt{x^2} = |x|} = \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$

always returns a (+)ve #

$\lim_{x\to(-\infty)} = \frac{\infty}{3(-\infty)} = -\frac{1}{3}$

Problem 10. Find the following limits, if they exists.

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(a) $\lim_{x\to\infty} \dfrac{4x^2-3x+5}{7-5x^2}$. $\approx \dfrac{4\cancel{x^2}}{-5\cancel{x^2}} = -\dfrac{4}{5}$ Ans.

$\dfrac{\#}{0} \to \pm\infty$

$\dfrac{0}{\#} = 0$

$\dfrac{0}{0}$ is indet.

3, $x=5$, 10

$$|5-x| = \begin{cases} (5-x), & x<5 \\ -(5-x), & x\geq 5 \end{cases}$$

$\dfrac{2(25)-10(5)}{|5-5|} \sim \dfrac{0}{0}$

(b) $\lim_{x\to 5} \dfrac{2x^2-10x}{|5-x|}$. $= \dfrac{2x(x-5)}{|5-x|}$

$\dfrac{(x-5)}{(5-x)} = \dfrac{(x-5)}{-(x-5)} = -1$

$5-x = -x+5$
$= -1(x-5)$

LHL: $\lim_{x\to 5^-} \dfrac{2x(x-5)}{(5-x)} = 2(5)(-1) = -10$

$\neq$

RHL $\lim_{x\to 5^+} \dfrac{2x(x-5)}{-(5-x)} = \dfrac{2x\cancel{(x-5)}}{\cancel{(x-5)}} = +10$

$\lim_{x\to 5}$ DNE Ans.

(c) $\lim_{x\to-\infty} \dfrac{7x-3}{\sqrt{4x^2+3x+1}}$. $\approx \dfrac{7x}{\sqrt{4x^2}}$

$\lim_{x\to(-\infty)} \dfrac{7x}{2|x|} = \dfrac{7\cancel{x}}{2(-\cancel{x})} = -\dfrac{7}{2}$ Ans.

$\lim_{x\to\infty} \dfrac{7x}{2|x|} = \dfrac{7x}{2(x)} = +\dfrac{7}{2}$

$\sqrt{4x^2+3x}$ vs $\sqrt{4x^2}+3x$

(d) $\lim_{x\to\infty} \arctan(e^x)$. $= \arctan(e^\infty)$

$= \arctan(\infty)$

$= \pi/2$ Ans.

Rem: $\arctan(-\infty)$
$= -\pi/2$

$= \pi/2$ Ans.

$= -\pi/2$

(e) $\lim_{x\to\infty} \dfrac{3e^{-2x}+e^{7x}}{5e^{-2x}-3e^{7x}}$.

$\lim_{x\to(-\infty)} e^{x} \to e^{-\infty} \Rightarrow 0$

$\lim_{x\to(-\infty)} e^{-x} \to e^{\infty} \to \infty$

$\sim \dfrac{3e^{-2x}}{5e^{-2x}}$

$= \dfrac{3}{5}$ Ans.

$\lim_{x\to\infty} \sim \dfrac{e^{7x}}{-3e^{7x}} = -\dfrac{1}{3}$

(f) $\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x^2}\right)+5\,.\right] = 0+5 = 5$ Ans.

$= \lim_{x\to 0} \underbrace{x^2\cos\left(\frac{1}{x^2}\right)}_{(0)\cos(\infty)} + \lim_{x\to 0}(5) \to 5$

$\leftarrow$ Squeeze theorem.

$\lim_{x\to 0}(-1)x^2 \le \lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right) \le \lim_{x\to 0}(+1)x^2$

$0 \le \underbrace{\lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right)}_{=0} \le 0$

8

$\ln A - \ln B = \ln\left(\frac{A}{B}\right)$

(g) $\lim_{x\to\infty}[\ln(x^3+6)-\ln(2x^3-1)]$ $\sim \ln(\infty)-\ln(\infty)$

$\sim \infty - \infty \leftarrow$ indeterminate

$\lim_{x\to\infty} \ln\left(\dfrac{x^3+6}{2x^3-1}\right)$

$\Rightarrow \lim_{x\to\infty}\left(\dfrac{x^3+6}{2x^3-1}\right) = \dfrac{1}{2}$

Ans: $\ln\left(\frac{1}{2}\right)$

(h) $\lim_{x\to\infty} \ln(5^x-3)$

∞

(h) $\lim_{x\to\infty} \ln(5^x - 3)$

$$\lim_{x\to\infty} (5^x) \sim 5^\infty \to \infty$$

Ans: $\ln(\infty) \to \infty$

Problem 11. Given that $(3x+2) \leq f(x) \leq (x^3+4)$ for $x \geq -2$, find $\lim_{x\to 1} f(x)$.

D: $[-2, \infty)$

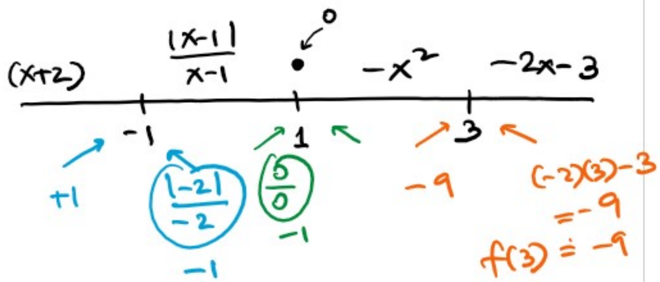
$$\lim_{x\to 1} (3x+2) \leq \lim_{x\to 1} f(x) \leq \lim_{x\to 1} (x^3+4)$$

$$5 \leq \lim_{x\to 1} f(x) \leq 5$$

$$\therefore \lim_{x\to 1} f(x) = 5$$

Problem 12. For what value(s) of x is $f(x)$ not continuous?

$$f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ \dfrac{|x-1|}{x-1} & \text{if } -1 < x \le 1 \\ \boxed{0 \quad \text{if } x = 1} & \\ -x^2 & \text{if } 1 \le x < 3 \\ -2x-3 & \text{if } x \ge 3 \end{cases}$$



LHL ≠ RHL $\therefore$ $f(x)$ has $\boxed{\text{a discont @ } x = -1}$

→ jump discont.

@ x = 1

LHL $\displaystyle \lim_{x \to 1^-} \frac{|x-1|}{x-1} = \frac{-(x-1)}{(x-1)} = -1$

RHL: 1^+

$-(1)^2 = -1$

LHL = RHL

$\neq f(1)$

$$|x-1| = \begin{cases} -(x-1), & x < 1 \\ +(x-1), & x \ge 1 \end{cases}$$

$$\left(\lim_{x \to 1} f(x) = -1 \right) \neq \left(f(1) = 0 \right)$$

$\therefore$ $f(x)$ also has a discont @ $x = 1$
$\Rightarrow$ hole.

Since $\displaystyle \lim_{x \to 3} f(x) = f(3) = -9$

$f(x)$ is cont. @ $x = 3$

Problem 13. Use the Intermediate Value Theorem to find an interval which contains the point of intersection of the functions $y = x^3 - 3x^2$ and $y = x - 5$.

$y_1 = x^3 - 3x^2 \qquad y_2 = x - 5$

$x^3 - 3x^2 = x - 5$

$x^3 - 3x^2 - x + 5 = 0$

$f(x) = x^3 - 3x^2 - x + 5 \leftarrow$ Solve for zero.

$f(0) = 5$

$$\boxed{\begin{aligned} f(1) &= 1 - 3 - 1 + 5 = 2 \\ f(2) &= 8 - 12 - 2 + 5 = -1 \end{aligned}}$$

$\rightarrow$ Intersect between $x = 1$ and $x = 2$

Problem 14. Find the values of a and b that would make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} x + 3a & \text{if } x \le 2 \\ ax^2 + bx + 2 & \text{if } 2 < x < 4 \\ 2bx - 2 & \text{if } x \ge 4 \end{cases}$$

$x + 3a \qquad ax^2 + bx + 2 \qquad 2bx - 2$

$2 \qquad 4$

① $x = 2$

$2 + 3a = 4a + 2b + 2$

$a + 2b = 0$

$a = -2b$

$a = -2\left(\frac{1}{9}\right)$

$= \frac{-2}{9}$ Ans.

② $x = 4$

$16a + 4b + 2 = 8b - 2$

$16a - 4b = -4$

$16(-2b) - 4b = -4$

$-32b - 4b = -4$

$-36b = -4$

$b = \frac{-4}{-36} = \frac{1}{9}$ Ans

Problem 15. Use the definition of the derivative to find $f'(x)$ for the function $f(x) = \sqrt{7+x}$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \sqrt{7+x}$$

$$f(x+h) = \sqrt{7+x+h}$$

$$(a-b)(a+b) = a^2 - b^2$$

$$f'(x) = \lim_{h\to 0} \left[\frac{\sqrt{7+x+h} - \sqrt{7+x}}{h}\right]\left[\frac{\sqrt{7+x+h} + \sqrt{7+x}}{\sqrt{7+x+h} + \sqrt{7+x}}\right]$$

$$= \lim_{h\to 0} \frac{(\cancel{7} + \cancel{x} + h) - (\cancel{7} + \cancel{x})}{h\left(\sqrt{7+x+h} + \sqrt{7+x}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{7+x+h} + \sqrt{7+x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{7+x+\cancel{h}^{\,0}} + \sqrt{7+x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{7+x}}}$$ Ans.